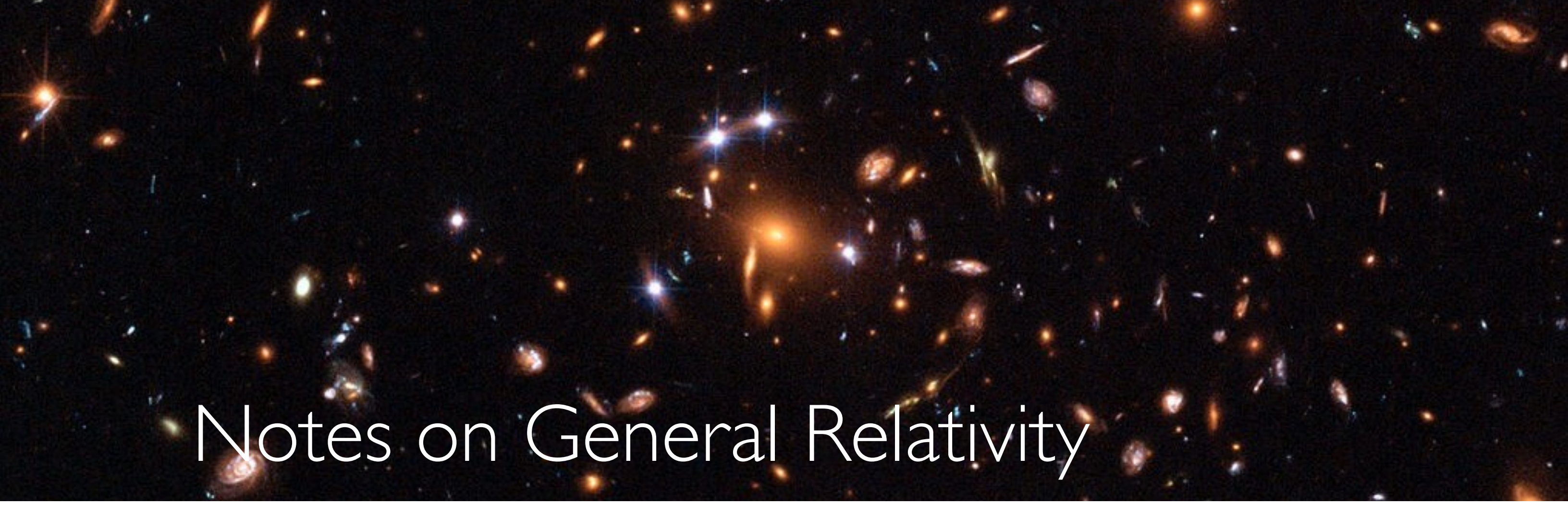
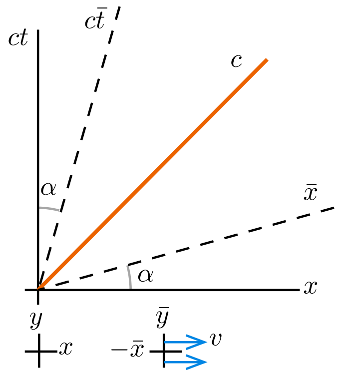


Notes on General Relativity

Special relativity



Postulates: the speed of light is constant for all reference frames and all frames of reference must produce equivalent physics results.

Reference frames: a reference frame (bar) moves with speed v in the positive x direction. In a ct vs x plot, velocity is represented by the slope, so $\tanh\alpha := \beta := v/c$. The line of c must be constant in all reference frames. "Squeezing" the axes towards c is the only possible transformation: we need to use hyperbolic trigonometry to decompose:

$$\begin{cases} \bar{x} = x\cosh\alpha - ct\sinh\alpha \\ c\bar{t} = ct\cosh\alpha - x\sinh\alpha \\ \bar{y} = y \\ \bar{z} = z \end{cases}$$

Hyperbolic identities: $\cosh^2\alpha - \sin^2\alpha = 1$, $\tanh\alpha = \beta \implies \sinh\alpha = \dfrac{\beta}{\sqrt{1-\beta^2}}$ and $\cosh\alpha = \dfrac{1}{\sqrt{1-\beta^2}}$.

Lorentz transformation (matrix form, showing only the affected coordinates):

$$\begin{bmatrix} c\bar{t} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

Dilation of time: differentiating the Lorentz transformation and if $dx = 0$, we get $d\bar{t} = \gamma dt$.

Contraction of length: stick with ends $\bar{x}_0 = x_0 = 0$, $\bar{x} = L_0$, measured at $t = 0$ in the laboratory frame. Then, $L_0 = x\gamma \implies x = L_0/\gamma$.

Momentum: Let τ be the time in the own frame of reference. We redefine physical quantities by using $t \to \tau$ (u, the speed of the object, also becomes v, the speed of the moving frame). Momentum becomes $\vec{\mathbf{p}} = m\dfrac{d\vec{\mathbf{x}}}{d\tau} = m\dfrac{d\vec{\mathbf{x}}}{dt}\dfrac{d\tau}{dt} = \gamma m\vec{\mathbf{u}}$. Without this modification, momentum is not conserved.

Work-Kinetic energy theorem: $W = K - \cancel{K_0}$; $W = \displaystyle\int \frac{d\vec{\mathbf{p}}}{dt}\cdot d\vec{\ell} = \int \frac{d}{dt}(\gamma m\vec{\mathbf{u}})\cdot\vec{\mathbf{u}}dt = m\int u\,d(\gamma u)$. Integrating by parts, $W = \gamma m u^2 - m\displaystyle\int \frac{u\,du}{\sqrt{1-u^2/c^2}} = \gamma m u^2 + mc^2\sqrt{1-u^2/c^2} - mc^2$; reordering,

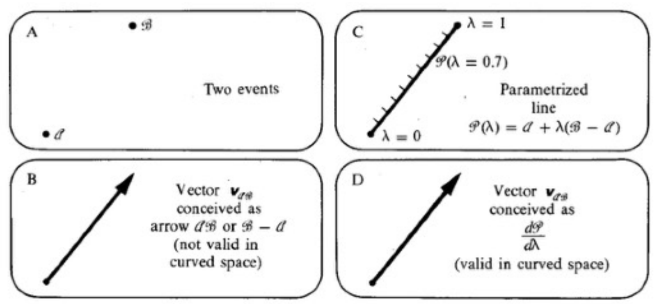
$\implies K = \gamma m c^2 - m c^2$. For small values of u, an expansion of γ yields
$K = mc^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots \right] - mc^2 = \frac{mu^2}{2}$, that is, the classical expression.

Energy-momentum relation: the total energy $E = K + mc^2$ is $E = \gamma mc^2$. $p^2 = (mc)^2/(1 - u^2/c^2)$ $\implies u^2/c^2 = p^2/(m^2c^2 + p^2)$. With that, we find $\gamma = \sqrt{\frac{m^2c^2 + p^2}{m^2c^2}}$, and $E = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}$. The energy-momentum relation is then $E^2 = p^2c^2 + m^2c^4$. Even if we started with the momentum of a massive particle, when we put $m = 0$, $E = pc \implies p = E/c$, which can also be found with classical electromagnetism.

Lagrangian: we know that $p_i = \frac{\partial L}{\partial u_i}$. Inserting the momentum, we get $L = -mc^2\sqrt{1 - \beta^2}$. We can add a velocity independent potential and the relation will remain the same.

Foundations

Definition of a vector: difference of two points. Then, tangent vector becomes the derivative of a point when the difference is infinitesimal. $\mathbf{v} = \frac{d\mathscr{P}}{d\lambda}$, where $\mathscr{P}(\lambda)$ is a parametrized line (composed of points).

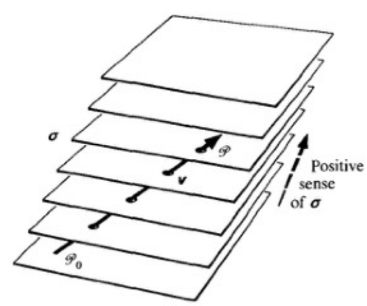


"4D position" (event) with coordinate description: $\mathscr{P} - \mathscr{O} = x^0\mathbf{e}_0 + x^1\mathbf{e}_1 + x^1\mathbf{e}_2 + x^3\mathbf{e}_3$ (event $\mathscr{P}$ respect to the origin $\mathscr{O}$). $\mathbf{e}_\mu$ are the vectors of an orthonormal basis for the manifold.

4-velocity of the particle in coordinate description: $\mathbf{u} = d\mathscr{P}/d\tau = (dx^\mu/d\tau)\mathbf{e}_\mu$.

A metric tensor is a machine with two slots for inserting vectors, and gives a real number such that $\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$, the scalar product of both vectors. It is linear. The metric coefficients are defined via the basis vectors, such that $g_{\mu\nu} = \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta)$.

A one-form (a.k.a. Pfaffian forms) can be imagined as a vector that pierces surfaces. The number of surfaces ϕ pierced can be denoted as $\langle \tilde{\mathbf{k}}, \mathbf{v} \rangle = \phi(\mathscr{P}) - \phi(\mathscr{P}_0)$. The surfaces are pierced in a specific direction. This corresponds to the gradient. This is why basis one-forms are denoted by $\underline{\omega}^\alpha = \mathbf{d}x^\alpha$, that is, the vector $\mathbf{e}_\alpha$ pierces one surface of the basis one-form $\underline{\omega}^\alpha$.



Components: since the bases are orthonormal (*dual*), the components of an arbitrary one-form $\mathbf{A} = A_\alpha \underline{\omega}^\alpha$ can be calculated as $\langle \mathbf{A}, \mathbf{e}_\alpha \rangle = A_\beta \langle \underline{\omega}^\beta, \mathbf{e}_\alpha \rangle = A_\beta \delta^\beta_\alpha = A_\alpha$, and the same for a vector. The directional derivative operator is $\partial_\mathbf{v} = v^\alpha \partial/\partial x^\alpha$ (by the chain rule).

Example for Schwarzschild: basis 1-forms:

$$\begin{cases} \underline{\omega}^0 = (1-2M/r)^{1/2}\mathbf{d}t \\ \underline{\omega}^1 = (1-2M/r)^{-1/2}\mathbf{d}r \\ \underline{\omega}^2 = r\,\mathbf{d}\theta \\ \underline{\omega}^3 = r\sin\theta\,\mathbf{d}\phi \end{cases}$$

Coordinate-free geometry

Differentiable manifold: a space such that any point has a neighborhood which is homeomorphic to the interior of the Euclidean unit ball (locally flat, continuous).

Chart: subset of the manifold that allows us to assign to every point, local coordinates.

Atlas: collection of compatible charts (those for which the overlapping is a homeomorphism, or one-to-one) that cover the manifold.

Tangent bundle: the set of all tangent spaces at points of the manifold.

Symmetries and operations

Dot product: $\mathbf{u}\cdot\mathbf{v} = \mathbf{g}(\mathbf{u},\mathbf{v}) = g_{\alpha\beta}u^\alpha v^\beta = u_\alpha v^\beta$. *Inner product:* $\langle \underline{\xi}, \mathbf{b}\rangle = \xi_\mu b^\mu$, $\langle$1-form, tang. vect.$\rangle$

Tensor, tensor product: putting vectors/one-forms side by side to form a higher-ranked object, a tensor. E.g., $\mathbf{S} = S^\alpha_{\ \beta}\mathbf{e}_\alpha \otimes \underline{\omega}^\beta$, and the components can be recovered with $S^\alpha_{\ \beta} = \mathbf{S}(\underline{\omega}^\alpha, \mathbf{e}_\beta)$, and partially with $\mathbf{S}(:\,,\mathbf{e}_\beta) = S^\alpha_{\ (\beta)}\mathbf{e}_\alpha = S_{\alpha(\beta)}\underline{\omega}^\alpha$. In general, $(\mathbf{u}\otimes\mathbf{v})(\underline{\sigma},\underline{\lambda}) = \langle\underline{\sigma},\mathbf{u}\rangle\langle\underline{\lambda},\mathbf{v}\rangle$.

Symmetrization: $V_{\mu\nu} = V_{\nu\mu} = V_{(\mu\nu)} = \dfrac{1}{2}(V_{\mu\nu} + V_{\nu\mu})$

Antisymmetrization: $V_{\mu\nu} = -\,V_{\nu\mu} = V_{[\mu\nu]} = \dfrac{1}{2}(V_{\mu\nu} - V_{\nu\mu})$

Levi-Civita tensor: The components of $\underline{\varepsilon} = \varepsilon(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots)$ give +1 for even permutations of 0,1,2,... and −1 for odd permutations, and 0 if two elements repeat, for a right-handed *basis* $\{\mathbf{e}_1, \mathbf{e}_2\}$. For $\underline{\varepsilon} = \varepsilon(\mathbf{e}_1, \mathbf{e}_2)$, it corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$.

Wedge product: antisymmetrization of the tensor product. (bivector) $\mathbf{u}\wedge\mathbf{v} = \mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}$ (vectors), and similarly for 1-forms (the result is a 2-form).

Wedge product for p*-forms*: distributive and associative, but commutation rule changes: $\underset{p}{\underline{\alpha}} \wedge \underset{q}{\underline{\beta}} = (-1)^{pq}\underset{q}{\underline{\beta}}\wedge\underset{p}{\underline{\alpha}}$, where $\underset{p}{\underline{\alpha}} = \underline{\sigma}\wedge\underline{\tau}\ldots$ (p factors).

Dual: ${}^\star J_{\alpha\beta\gamma} = \dfrac{1}{1!}J^\mu\varepsilon_{\mu\alpha\beta\gamma}$. It takes an antisymmetric *p*-rank tensor and creates a new, 4-*p* rank tensor for which ${}^{\star\star}\mathbf{J} = (-1)^{p-1}\mathbf{J}$ (for a vector, ${}^{\star\star}\mathbf{J} = \mathbf{J}$). This *dual* is different from the idea that two sets of vectors and 1-forms are dual to each other. For a 2-rank tensor, ${}^\star F_{\alpha\beta} = \frac{1}{2}F^{\mu\nu}\varepsilon_{\mu\nu\alpha\beta}$.

Examples: $\mathbf{B} = \overrightarrow{B} = (B_x, B_y, B_z) \to {}^\star\mathbf{B} = \mathsf{B} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix}$

$$\underline{\omega} = \vec{\omega} = (\omega_x, \omega_y, \omega_z) \rightarrow {}^{\star}\underline{\omega} = \overleftrightarrow{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

Musical isomorphism: $\mathbf{A}^\flat = A_\mu \mathbf{d}x^\mu$, $\mathbf{A}^\sharp = A^\mu \mathbf{e}_\mu$

Exterior derivative: the operator d generates a 1-form from a 0-form f: $\mathbf{d}f = f_{,i}\mathbf{d}x^i$. For p-forms, it generates a $(p+1)$-form such that the operator is linear, and $\mathbf{d}\left(\underset{p}{\underline{\alpha}} \wedge \underset{q}{\underline{\beta}}\right) = \mathbf{d}\underset{p}{\underline{\alpha}} \wedge \underset{q}{\underline{\beta}} + (-1)^p \underset{p}{\underline{\alpha}} \wedge \mathbf{d}\underset{q}{\underline{\beta}}$, also, the operator satisfies the property that $\mathbf{d}(\mathbf{d}f) = 0$.

The exterior derivative of a 1 form is the 2-form $\underline{\xi} = \mathbf{d}\underline{\beta} = \dfrac{\partial \beta_\alpha}{\partial x^\mu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\alpha$ (there is a factor of $1/1!$).

Integration of p-forms: example: integration for 1-form along a curve:

$$\int \mathbf{d}f = \int_{\mathscr{P}(0)}^{\mathscr{P}(1)} \left\langle \mathbf{d}f, \frac{\mathscr{P}(\lambda)}{d\lambda} \right\rangle d\lambda = \int_{\mathscr{P}(0)}^{\mathscr{P}(1)} \frac{df}{d\lambda} d\lambda = f[\mathscr{P}(1)] - f[\mathscr{P}(0)]$$

Generalized Stokes theorem: $\int_{\mathscr{V}} \mathbf{d}\underline{\sigma} = \int_{\partial\mathscr{V}} \underline{\sigma}$. Example I (previous line, integration of p-forms, since f is a 0-form). Example II: $\int_{\mathscr{V}} \mathbf{dv} = \int_{\partial\mathscr{V}} \mathbf{v} \implies \int_{\mathscr{V}} (\vec{\nabla} \times \vec{v}) \cdot \mathbf{d}\vec{S} = \int_{\partial\mathscr{V}} \vec{v} \cdot \mathbf{d}\vec{l}$.

Gradient and divergence: (for scalars, $\underline{\nabla} f \equiv \mathbf{d}f$) In Minkowski, the components of $\underline{\nabla}\mathbf{S}$ are the partial derivatives of the components of $\mathbf{S}$, e.g., $\partial_\delta S_{\alpha\beta\gamma} = S_{\alpha\beta\gamma,\delta}$ (rank increases by one). The divergence $\underline{\nabla} \cdot \mathbf{S}$ on the first slot of $\mathbf{S}$ has components $S^{\alpha}{}_{\beta\gamma,\alpha}$. Also, note that the divergence of the dual becomes

$$\begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix} = \begin{pmatrix} \partial_y B_z - \partial_z B_y & -\partial_x B_z + \partial_z B_x & \partial_x B_y - \partial_y B_x \end{pmatrix}$$ that is, $\vec{\nabla} \cdot \mathsf{B} = \vec{\nabla} \times \vec{B}$. Or, $\vec{\nabla} \cdot \mathsf{B} = -\mathsf{B} \cdot \overleftarrow{\nabla} = \vec{\nabla} \times \vec{B}$.

Electromagnetism

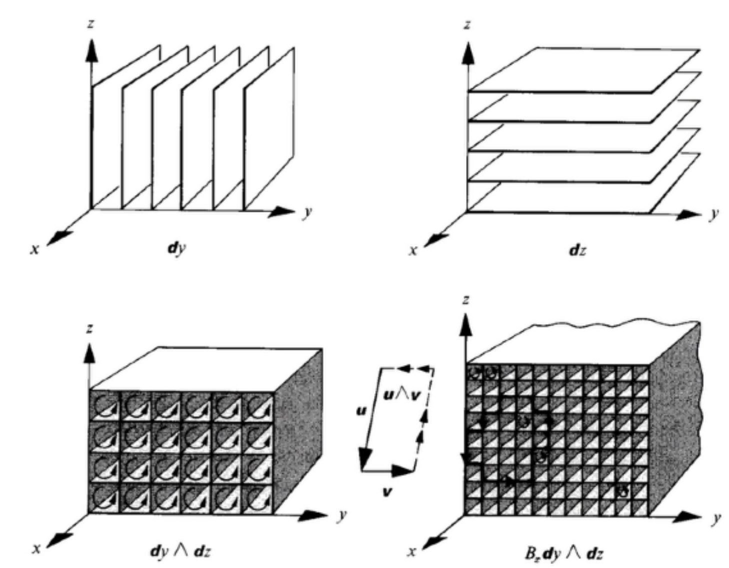
Faraday tensor: antisymmetric 2-form: $\mathbf{F} = \frac{1}{2}F_{\alpha\beta}\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta (= F_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta)$. Example: only magnetic field in x direction: $F_{yz} = -F_{zy} = B_x \implies \mathbf{F} = B_x \mathbf{d}y \wedge \mathbf{d}z$.

In general in matrix form, for Minkowski sign. +2,

$$\mathbf{F} = \mathbf{F}_{\flat\flat} = (F_{\alpha\beta}) = \begin{pmatrix} 0 & -\vec{E}^T \\ \vec{E} & -\mathsf{B} \end{pmatrix}$$ or

$$\mathbf{F} = \mathbf{F}^{\sharp\sharp} = (F^{\alpha\beta}) = (\eta_{\mu\alpha}F^{\mu\nu}\eta_{\nu\beta}) = \begin{pmatrix} 0 & \vec{E}^T \\ -\vec{E} & -\mathsf{B} \end{pmatrix}$$

A 2-form is constructed by the intersection of the planes that define the 1-forms. It gives a sense of *circulation* in a direction (orientation).



Lorentz force: $\dot{\mathbf{p}} = \dfrac{d\mathbf{p}}{d\tau} = e\mathbf{F}(:\ ,\mathbf{u}) = e\langle \mathbf{F}, \mathbf{u}\rangle$, where $\mathbf{u} = \dfrac{dx^\alpha}{d\tau}\mathbf{e}_\alpha$. Example: for a magnetic field in the x direction,
$\dot{p}_\alpha \mathbf{d}x^\alpha = eB_x\langle \mathbf{d}y \wedge \mathbf{d}z,\ \mathbf{u}\rangle = eB_x(\mathbf{d}y\langle \mathbf{d}z, \mathbf{u}\rangle - \mathbf{d}z\langle \mathbf{d}y, \mathbf{u}\rangle)$
$= eB_x u^z \mathbf{d}y - eB_x u^y \mathbf{d}z$ (generalized BAC - CAB rule, one can easily demonstrate it using the definition of the wedge product with the tensor product).

Maxwell equations:

1. $\underline{\nabla} \cdot \mathbf{F} = 4\pi \mathbf{J}$ (also $\mathbf{d}^\star\mathbf{F} = 4\pi{}^\star\mathbf{J}$) implies ($\mathbf{F}^{\sharp\sharp} \cdot \underset{\leftarrow}{\nabla} = 4\pi\mathbf{J}^\sharp$)
$$\begin{pmatrix} 0 & \vec{E}^T \\ -\vec{E} & -\mathsf{B} \end{pmatrix}\begin{pmatrix} \overleftarrow{\partial_t} \\ \overleftarrow{\nabla} \end{pmatrix} = \begin{pmatrix} \vec{\nabla}\cdot\vec{E} \\ -\partial_t\vec{E} - \mathsf{B}\cdot\overleftarrow{\nabla} \end{pmatrix} = \begin{pmatrix} \vec{\nabla}\cdot\vec{E} \\ -\partial_t\vec{E} + \vec{\nabla}\times\vec{B} \end{pmatrix} = 4\pi\begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$$
2. $\mathbf{dF} = 0$ or $\underline{\nabla} \cdot {}^\star\mathbf{F} = 0$, where ${}^\star\mathbf{F}$ is the *Maxwell* tensor, which is obtained by replacing $\vec{E} \to \vec{B}$ and $-\mathsf{B} \to \mathsf{E}$.

Wave equation: using that the exterior derivative satisfies $\mathbf{dd} = 0$, one can define the vector potential $\mathbf{A}$ such that $\mathbf{F} = \mathbf{dA}$, and so, $\mathbf{ddA} = 0$ (Lorentz gauge condition). Also, $\mathbf{d}^\star\mathbf{dA} = 4\pi{}^\star\mathbf{J} \implies {}^\star\mathbf{d}^\star\mathbf{dA} = 4\pi\mathbf{J}$ and it may be shown that $\implies {}^\star\mathbf{d}^\star\mathbf{dA} = -\Box\mathbf{A}$ (the wave operator).

Continuity equation: From the first Maxwell equation, $\mathbf{d}^\star\mathbf{F} = 4\pi{}^\star\mathbf{J}$, we see that $\mathbf{dd}^\star\mathbf{F} = 4\pi\mathbf{d}^\star\mathbf{J} = 0$ (since $\mathbf{dd} = 0$), and then, $\mathbf{d}^\star\mathbf{J} = \underline{\nabla}\cdot\mathbf{J} = 0$.

Stress-energy tensor

It has two slots, and contains the information about energy density, momentum density, and stress (pressure).

Properties: symmetric, traceless (the photon is massless $\implies T^\alpha_\alpha = 0$), $T^{00} \geq 0$.

Interactions with vectors: if $\mathbf{u}$ is the 4-velocity of the observer, $\mathbf{T}(\mathbf{u},:)$ is momentum/volume, and $\mathbf{T}(\mathbf{u},\mathbf{u})$ is mass-energy/volume.

Conservation of energy-momentum: $\underline{\nabla}\cdot\mathbf{T} = 0$

Perfect fluid: $\mathbf{T} = (\rho + p)\mathbf{u}\otimes\mathbf{u} + \rho\mathbf{g}$. In the rest frame of the fluid, $(u^\alpha) = (1,0,0,0)$, and
$$(T_{\alpha\beta}) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Electromagnetic field: $4\pi T^{\mu\nu} = F^{\mu\alpha}F^\nu_\alpha - \dfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$. In matrix form, symbolically,
$\mathbf{T} = \begin{pmatrix} \text{energy density} & \vec{S}^T \\ \vec{S} & \text{Maxwell stress tensor} \end{pmatrix}$, where $\vec{S}$ is the Poynting vector; the energy density $= (\vec{E}^2 + \vec{B}^2)/(8\pi)$ is obtained from the work done by the field, and the Maxwell stress tensor is obtained by inserting Maxwell's equations into the Lorentz force.

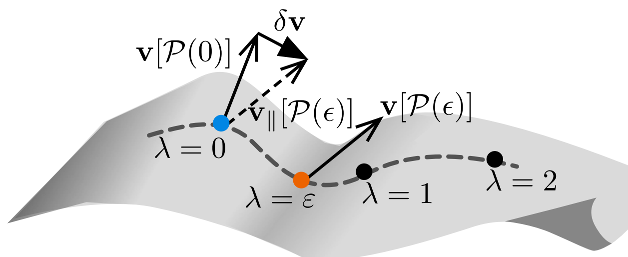
Geodesics and curved spacetime

Differential topology: results independent of the metric.

Vectors as operators: there is an isomorphism between $\mathbf{u} = e^\alpha \mathbf{e}_\alpha$ and $\partial_\mathbf{u} = u^\alpha \partial_\alpha$: both have the same components. This allows us to define a vector as the directional derivative: $\mathbf{u} = \partial_\mathbf{u}$. $\langle \mathbf{d}f, \mathbf{u} \rangle = \partial_\mathbf{u} f = \mathbf{u}[f]$

Commutators: $[\mathbf{u}, \mathbf{v}][f] = \mathbf{u}\{\mathbf{v}[f]\} - \mathbf{v}\{\mathbf{u}[f]\}$

Commutator of two vector fields: for a given basis, $[\mathbf{e}_a, \mathbf{e}_b] = D^c_{ab}\mathbf{e}_c$, where D^c_{ab} are the coefficients of the commutator. Commutators satisfy the Jacobi identity:
$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0$.



Lie derivative: $\mathcal{L}_\mathbf{u}\mathbf{v} = [\mathbf{u}, \mathbf{v}]$. It transports a vector field $\mathbf{v}$ along a vector field $\mathbf{u}$.

Covariant derivative: parallel transport of the vector $\mathbf{v}$ evaluated at $\mathscr{P}(\epsilon)$ to the point $\mathscr{P}(0)$ where the covariant derivative $\underline{\nabla}_\mathbf{u}\mathbf{v}$ is to be evaluated. The difference of the two vectors, $\delta\mathbf{v} = \mathbf{v}_{||}[\mathscr{P}(\epsilon)] - \mathbf{v}[\mathscr{P}(0)]$, in the limit $\epsilon \to 0$ is the covariant derivative.

Connection coefficients: $\Gamma^\alpha_{\beta\gamma} = \langle \underline{\omega}^\alpha, \underline{\nabla}_\gamma \mathbf{e}_\beta \rangle$ "α component of change in $\mathbf{e}_\beta$, relative to parallel transport, along $\mathbf{e}_\gamma$". *Warning!* This definition is not necessarily symmetrical in β, γ; it's *not* the Levi–Civita connection (coord. basis)

Chain rule: $\underline{\nabla}_\mathbf{u}(f\mathbf{A} \otimes \mathbf{B}) = (\underline{\nabla}_\mathbf{u} f)\mathbf{A} \otimes \mathbf{B} + f(\underline{\nabla}_\mathbf{u}\mathbf{A}) \otimes \mathbf{B} + f\mathbf{A} \otimes (\underline{\nabla}_\mathbf{u}\mathbf{B})$. If $f \to T^\alpha_\beta$ and $\mathbf{A}, \mathbf{B}, \mathbf{u}$ are the unit vectors and 1-forms, for which $\underline{\nabla}_{\mathbf{e}_\gamma}\mathbf{e}_\beta = \Gamma^\mu_{\beta\gamma}\mathbf{e}_\mu$, one can factor out the components of the covariant derivative in terms of the connection coefficients. The first term, $\underline{\nabla}_\mu f = \partial_\mu f$, is the common partial derivative, and the other terms are the correction for the curvature (well-known formula for the covariant derivative in terms of the Christoffel symbols)

Connection coefficients in terms of the metric: $\underline{\nabla}\mathbf{g} = 0 \implies g_{\alpha\beta;\gamma} = \dfrac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\mu_{\alpha\gamma}g_{\mu\beta} - \Gamma^\mu_{\beta\gamma}g_{\alpha\mu} = 0$. This implies the well-known formula for the Christoffel symbols with the metric.

Exterior derivative, equivalence principle: the partial derivatives get replaced with covariant derivatives in curved spacetime.

Geodesics and geodesic equation: $\underline{\nabla}_\mathbf{u}\mathbf{u} = 0 \implies u^\alpha{}_{;\beta}u^\beta = 0$ (the straightest line). The geodesic equation is, then, $(u^\alpha{}_{,\beta} + \Gamma^\alpha_{\gamma\beta}u^\gamma)u^\beta = 0$, where $u^\alpha = dx^\alpha/d\lambda$, with λ an affine parameter, that is, a parametrization of the curve.

Vector potential in curved spacetime: The exterior derivative in a curved spacetime is
$\mathbf{d}\mathbf{A} = A_{\mu;\nu}\mathbf{d}x^\nu \wedge \mathbf{d}x^\mu = A_{\mu;\nu}(\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu - \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu) = A_{\mu;\nu}\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu - A_{\mu;\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$ changing the dummy indexes in the second term, $A_{\mu;\nu}\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu - A_{\nu;\mu}\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu = (A_{\mu;\nu} - A_{\nu;\mu})\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu$
$(A_{\mu,\nu} + \Gamma^\xi_{\mu\nu}A_\xi - A_{\nu,\mu} - \Gamma^\xi_{\nu\mu}A_\xi)\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu$, but since the connection is symmetric in its lower indexes (torsion-free), the $\Gamma^\xi_{\mu\nu}$ cancel out, and then, $(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu = F_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$.

Maxwell equations: (1) the equation $\underline{\nabla} \cdot \mathbf{F} = 4\pi \mathbf{J}$ remains unchanged except that $\underline{\nabla}$ is now the covariant derivative (one index is then contracted). (2) The equation $\mathbf{dF} = 0$ remains exactly the same (only partial derivatives).

Proof and component forms: Second equation: $\mathbf{dF} = \dfrac{1}{3!} F_{\alpha\beta;\gamma} \mathbf{d}x^\gamma \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$.Expanding $(\gamma \to \alpha \to \beta)$,
$= F_{\alpha\beta;\gamma}(\mathbf{d}x^\gamma \otimes \mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta + \mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta \otimes \mathbf{d}x^\gamma + \mathbf{d}x^\beta \otimes \mathbf{d}x^\gamma \otimes \mathbf{d}x^\alpha$
$- \mathbf{d}x^\gamma \otimes \mathbf{d}x^\beta \otimes \mathbf{d}x^\alpha - \mathbf{d}x^\beta \otimes \mathbf{d}x^\alpha \otimes \mathbf{d}x^\gamma - \mathbf{d}x^\alpha \otimes \mathbf{d}x^\gamma \otimes \mathbf{d}x^\beta) = 0$.
Now, we change dummy indexes so we can collect the basis 1-forms:
$0 = \mathbf{d}x^\gamma \otimes \mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta (F_{\alpha\beta;\gamma} + F_{\gamma\alpha;\beta} + F_{\beta\gamma;\alpha} - F_{\beta\alpha;\gamma} - F_{\alpha\gamma;\beta} - F_{\gamma\beta;\alpha}) \implies 2F_{\alpha\beta;\gamma} + 2F_{\gamma\alpha;\beta} + 2F_{\beta\gamma;\alpha} = 0$ (using the antisymmetry of the Faraday tensor). Now, canceling the 2 and substituting the connection coefficients for the covariant derivatives,
$F_{\alpha\beta,\gamma} - \underline{\Gamma^\mu_{\alpha\gamma} F_{\mu\beta}}_{(1)} - \underline{\Gamma^\mu_{\beta\gamma} F_{\alpha\mu}}_{(2)} + F_{\gamma\alpha,\beta} - \underline{\Gamma^\mu_{\beta\alpha} F_{\mu\gamma}}_{(3)} - \underline{\Gamma^\mu_{\gamma\alpha} F_{\beta\mu}}_{(1)} + F_{\beta\gamma,\alpha} - \underline{\Gamma^\mu_{\gamma\beta} F_{\mu\alpha}}_{(2)} - \underline{\Gamma^\mu_{\alpha\beta} F_{\gamma\mu}}_{(3)} = 0$. The terms highlighted cancel out on using the antisymmetry of the Faraday tensor. Then, the components of the second equation read $F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0$. For the first equation, the situation is much simpler: ${F^{\alpha\beta}}_{;\beta} = 4\pi J^\alpha$.

Basics of curved spacetime (component form)

Parallel transport: $\mathbf{A} \to \mathbf{A} + \delta\mathbf{A}$. The change δA^i should be proportional to the vector itself and to the displacement: $\delta A^i = -\Gamma^i_{kl} A^k dx^l$, and compensates for the changes in the coordinates, so that the vector remains parallel to itself.

Derivative of a vector: $\dfrac{DA^i}{Dx^l} = \lim_{dx^l \to 0} \dfrac{A^i(x^l + dx^l) - [A^i(x^l) + \delta A^i]}{dx^l}$, but $A^i(x^l + dx^l) = A^i(x^l) + \dfrac{\partial A^i}{\partial x^l} dx^l$, and so, $\dfrac{DA^i}{Dx^l} = \dfrac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k$.

Derivative of a covector: $\dfrac{DA_i}{Dx^l} = \dfrac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k$

Symmetry of the Christoffel symbol: Let $A_i = \dfrac{\partial V}{\partial x^i} \implies \dfrac{DA_i}{Dx^k} - \dfrac{DA_k}{Dx^i} = (\Gamma^l_{ki} - \Gamma^l_{ik}) \dfrac{\partial V}{\partial x^l} = 0$ in cart. coordinates (the order of the derivative does not matter when substituting A_i). The left hand side is a tensor, so it must transform as a tensor and render 0 for any frame and coordinates. This implies that the lower indexes of the Christoffel symbol are symmetric. (Torsion-free connection)

Covariant derivative of the metric: $\dfrac{DA_i}{Dx^l} = \dfrac{D(g_{ik}A^k)}{Dx^l} = g_{ik} \dfrac{DA^k}{Dx^l} + A^k \dfrac{Dg_{ik}}{Dx^l}$. But since A_i and A^i are just representations of the same object, their derivatives should be related by the metric $\dfrac{DA_i}{Dx^l} = g_{ik} \dfrac{DA^k}{Dx^l}$ which implies $\dfrac{Dg_{ik}}{Dx^l} = 0$.

Components of the Christoffel symbol: The covariant derivative of the metric implies $\dfrac{\partial g_{ik}}{\partial x^l} = \Gamma^m_{kl} g_{im} + \Gamma^m_{il} g_{mk}$ (1). Permuting the symbols, we get $\dfrac{\partial g_{il}}{\partial x^k} = \Gamma^m_{ik} g_{lm} + \Gamma^m_{lk} g_{mi}$ (2) and $\dfrac{\partial g_{kl}}{\partial x^i} = \Gamma^m_{li} g_{km} + \Gamma^m_{ki} g_{ml}$ (3). Adding (2) and (3) and subtracting (1) we get $2\Gamma^n_{ik} g_{ln} = \dfrac{\partial g_{li}}{\partial x^k} + \dfrac{\partial g_{kl}}{\partial x^i} - \dfrac{\partial g_{ik}}{\partial x^l}$, and finally, $\Gamma^m_{ik} = \dfrac{1}{2} g^{ml} \left(\dfrac{\partial g_{li}}{\partial x^k} + \dfrac{\partial g_{kl}}{\partial x^i} - \dfrac{\partial g_{ik}}{\partial x^l} \right)$. This is the *Levi-Civita* connection.

Geodesic equations: $s = \int \sqrt{g_{ik} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda}} d\lambda$; $\delta s = 0 \implies \frac{d^2 x^m}{d\lambda^2} = -\Gamma^m_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda}$.

Curvature: it is measured by parallel transport of a vector in a closed path: $\Delta A_k = \oint_{\mathscr{C}} \Gamma^i_{kl} A_i dx^l$. If $\Delta A_k = 0$, the spacetime is flat. We apply Stokes' theorem $\oint_{\partial \mathscr{S}} A_i dx^i = \frac{1}{2} \int_{\mathscr{S}} df^{ik} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right)$ (where df^{ik} is the differential surface vector). Then, for the curvature integral, $\Delta A_k = \frac{1}{2} \int df^{ik} \left[\frac{\partial(\Gamma^i_{km} A_i)}{\partial x^i} - \frac{\partial(\Gamma^i_{kl} A_i)}{\partial x^k} \right]$. Conducting the chain rule and recognizing that the vector does not change position at the end of the trajectory, and then any change must come from $\delta A_i = \Gamma^k_{il} A_k dx^l$ which implies $\partial A_i / \partial x^l = \Gamma^n_{il} A_n$, then, we get $\Delta A_k = \frac{1}{2} \int \left[\frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^n_{km} \Gamma^i_{nl} - \Gamma^n_{kl} \Gamma^i_{nm} \right] A_i df^{lm}$. We call the term inside the parentheses the *Riemman* tensor, R^i_{klm}.

Ricci: $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$; *Curvature scalar*: $R = g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu$

Einstein field equations

Einstein tensor: If spacetime curves itself with the presence of matter-energy, $\mathbf{T}$ has to be proportional to a tensor $\mathbf{G}$ that contains information on how spacetime is deformed ($\mathbf{G} = \kappa \mathbf{T}$). Then, $\mathbf{G}$, just as $\mathbf{T}$, must comply with $\underline{\nabla} \cdot \mathbf{G} = 0$. Curvature is expressed by the Riemann tensor, $\mathbf{R}$ (4-rank). In analogy to Maxwell's first law, we take $\overline{\mathbf{G}} = {}^\star\mathbf{R}^\star$, the double dual of *Riemann* (two pairs of alternating indexes, two places to take the dual). But $\overline{\mathbf{G}}$ is a 4-rank tensor, so we reduce two indexes to form $\mathbf{G} = \overline{\mathbf{G}}(\underline{\omega}^\alpha, :, \mathbf{e}_\alpha, :)$, which indeed complies with $\underline{\nabla} \cdot \mathbf{G} = 0$.

Einstein field equations: Reducing everything to the limit of Newtonian gravity, we find $\kappa = 8\pi$ (geometrized units) or $\kappa = 8\pi G / c^4$ (SI units, G is the gravitational constant, not the trace of $\mathbf{G}$). So, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

Newtonian limit

Correspondence principle for the potentials: With the Lagrangian of special relativity with a Newtonian potential Φ, we have an action $I = -\int \left[mc^2 \sqrt{1 - \frac{v^2}{c^2}} + m\Phi \right] dt$. For $\Phi \ll c^2$ and $v \ll c$, we can put everything inside the square root with the expansion $\sqrt{1+x} \approx 1 + x/2$ ($x \to 2\Phi$):

$$I \approx -\int mc^2 \sqrt{1 - \frac{v^2}{c^2} + \frac{2\Phi}{c^2}} dt \approx -mc \int \sqrt{\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - v^2 dt^2}$$

Since $v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2}$ and $I = \int ds$, we identify the metric

$$ds^2 = -c^2 d\tau^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + dx^2 + dy^2 + dz^2 \text{ and then, } g_{00} = -1 - 2\Phi/c^2.$$

We could also have found the same result by starting from the geodesic equations and saying that the left-hand side is the acceleration (then, by Newton's second law, it is equal to the potential). The right-hand side is simplified by the linear approximation $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ ($\eta_{\mu\nu}$: Minkowski); the Christoffel symbols reduce to only $\Gamma^i_{00} = h_{00,i}/2$, and the whole equation yields $\Phi_{,i} = h_{00,i}c^2/2$ $\implies g_{00} = -1 - 2\Phi/c^2$.

Christoffel symbols: for the approximate metric, $\dfrac{\partial g_{00}}{\partial x^\alpha} = -\dfrac{2}{c^2}\dfrac{\partial \Phi}{\partial x^\alpha}$. Using this, we calculate $\Gamma^\alpha_{00} = \dfrac{1}{c^2}\dfrac{\partial \Phi}{\partial x^\alpha}$, neglecting quadratic terms.

Proportionality constant in Einstein Field Equations: for the time component of the equations, $R^0_0 = \kappa\left(T^0_0 - \dfrac{1}{2}T\right)$. But for $p \to 0$, $T \approx T^0_0 = -\rho c^2$, which implies $R^0_0 = -\dfrac{1}{2}\kappa\rho c^2$. Now, $R_{00} = \dfrac{\partial \Gamma^\alpha_{00}}{\partial x^\alpha} = \dfrac{\nabla^2\Phi}{2}$ (To lower the indices in the right-hand side, we just multiply by $g^{00} \approx -1$, since we are ignoring quadratic terms). Then, $\nabla^2\Phi = \kappa c^4\rho/2$. Comparing with the Newtonian Gauss' law, $\nabla^2\Phi = 4\pi G\rho$, we get $\kappa = 8\pi G/c^4$.

Linearized theory of relativity

It is based in the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$. The information of the curvature is in $h_{\mu\nu}$, and $\eta_{\mu\nu}$ is used to raise and lower indexes. Christoffel: $\Gamma^\mu_{\alpha\beta} = \dfrac{1}{2}(h_\alpha{}^\mu{}_{,\beta} + h_\beta{}^\mu{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu})$. Ricci lineariz.: $R_{\mu\nu} \approx \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu}$; $R \approx \eta^{\mu\nu}R_{\mu\nu}$; $h = h^\alpha_\alpha$.

Einstein field equations:

$$\underbrace{h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu}}_{2\ \textbf{Ricci}} - \underbrace{\eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - h^{\beta}_{,\beta})}_{\textbf{metric} \times R} = 16\pi T_{\mu\nu}$$

which can be simplified using the transformations $\bar{h}_{\mu\nu} = h_{\mu\nu} - \dfrac{1}{2}\eta_{\mu\nu}h$ and $h_{\mu\nu} = \bar{h}_{\mu\nu} - \dfrac{1}{2}\eta_{\mu\nu}\bar{h}$ to $-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} + \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{,\nu} + \bar{h}_{\nu\alpha}{}^{,\alpha}{}_{,\mu} = 16\pi T_{\mu\nu}$.

Gauge condition and wave equation: One can impose the gauge condition $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ (analog of the Lorentz gauge of electromagnetism), and, after that, we get $-h_{\mu\nu,\alpha}{}^{\alpha} = 16\pi T_{\mu\nu}$, which is a wave equation, and has a wave-like solution.

Nearly Newtonian gravitational fields: the correction has the form of a retarded potential, $\bar{h}_{\mu\nu}(t, \vec{x}) = \displaystyle\int \dfrac{4T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}d^3x'$. For a nearly Newtonian source, and if the retardation is negligible, $\bar{h}_{00} = -4\Phi$ (the rest is zero), with Φ being the Newtonian potential, and the metric is $ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$. For points far from the source, $\Phi \approx -M/r$.

[The Newtonian potential with quadrupole moment is $\Phi = -\dfrac{GM}{r} - C\dfrac{GM}{r^3}\dfrac{1}{2}(1 - 3\cos^2\theta)$. Look up *geopotential*.]

Geometric optics

The fundamental laws of geometric optics are: (1) light rays are null geodesics; (2) the polarization vector is perpendicular to the rays and is parallel-propagated along the rays; (3) the amplitude is governed by an adiabatic invariant which, in quantum language, states that the number of photons is conserved.

Validity of geometric optics: (wavelength→) $\lambda \ll \mathscr{R}$ (←radius of curvature of spacetime)

Description with the vector potential: $\mathbf{A} = \Re\{\text{amplitude} \times e^{i\theta}\}$, where $\theta \sim k \times$(distance propagated) is the *phase*. [θ is analog to S in quantum mechanics]

Wave vector: since $\theta = k_\alpha x^\alpha + \text{const}$, this means that $\theta = \mathbf{k} \cdot \mathbf{x} + \text{const}$, or by a Taylor expansion, $\theta = \underline{\nabla}\theta \cdot \mathbf{x} +$ other terms, and so, we identify $\underline{\nabla}\theta = \mathbf{k}$. By the correspondence principle, the phase should be equal to $\theta = \overrightarrow{k} \cdot \overrightarrow{x} - \omega t$, so, $k^0 = \omega$ and $|\overrightarrow{k}| = \sqrt{k^i k_i} = 2\pi/\lambda = \omega$ (geom. units).

Amplitude: amplitude = [indep. of λ] + [small corrections] that is, amplitude $= \mathbf{a} + \epsilon\mathbf{b} + \epsilon^2\mathbf{c} + \cdots$, where $\mathbf{b} \propto \lambda$, $\mathbf{c} \propto \lambda^2$, and $\epsilon = 1$ (inserted just to keep track of the order).

Polarization: to first order, $\mathbf{a} = a\mathbf{f}$, where a is the scalar amplitude $a = (\mathbf{a} \cdot \bar{\mathbf{a}})^{1/2}$ and $\mathbf{f}$ is the polarization. For example, $\mathbf{f} = \mathbf{e}_x$ is linear polarization in the x direction, and $\mathbf{f} = \dfrac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y)$ is righthand circular polarization.

Electromagnetic field tensor: it's always perpendicular to the rays, $\mathbf{F} \cdot \mathbf{k} = 0$, so $\mathbf{F} = \mathbf{dA} = \Re\{i a e^{i\theta}\mathbf{k} \wedge \mathbf{f}\}$ $\implies F_{\alpha\beta} = a_{\alpha\beta}e^{i\theta/\epsilon} + \cdots$ (omitting the real sign).

Geodesic equation: Introducing the field tensor in the second Maxwell equation, we have $\theta_{,\gamma}a_{\alpha\beta} + \theta_{,\alpha}a_{\gamma\beta} + \theta_{,\beta}a_{\alpha\gamma} = 0 \implies k_\gamma a_{\alpha\beta} + k_\alpha a_{\gamma\beta} + k_\beta a_{\alpha\gamma} = 0$ (a). In the first Maxwell equation, we substitute $F^{\alpha\beta} = e^{i\theta}(a^{\alpha\beta} + \cdots)$ and get $F^{\alpha\beta}{}_{;\beta} = e^{i\theta/\epsilon} i \dfrac{\theta_{,\beta}}{\epsilon} a^{\alpha\beta} +$ connection terms of order $\epsilon^0 + \cdots = 0$. Collecting the terms with order $1/\epsilon$ and substituting $\theta_{,\beta} = k_\beta$, we get $a^{\alpha\beta}k_\beta = a_{\alpha\beta}k^\beta = 0$ (b). Now, multiplying (a) by k^γ we get $a_{\alpha\beta}k_\gamma k^\gamma + a_{\gamma\beta}k_\alpha k^\gamma + a_{\alpha\gamma}k_\beta k^\gamma = 0$. Using (b), we get $a_{\alpha\beta}k_\gamma k^\gamma = 0$, and, since in general $a_{\alpha\beta} \neq 0$, $k_\gamma k^\gamma = 0$, that is, the condition that geodesics must be null. Let's remember that $k_{\gamma;\beta} = k_{\beta;\gamma}$, since $\mathbf{k}$ is a gradient. Now, from the null geodesic condition, let us form a contravariant tensor with the derivative, $0 = g^{\alpha\gamma}(k^\beta k_\beta)_{;\gamma} = 2g^{\alpha\gamma}k^\beta k_{\beta;\gamma}$ we get $2g^{\alpha\gamma}k^\beta k_{\gamma;\beta} = 2k^\beta k^\alpha_{;\beta} = 0 \implies k^\beta k^\alpha_{;\beta} = 0$, which is the geodesic equation.

Light rays: they are defined as curves $\mathscr{P}(l)$ normal to surfaces of constant phase θ. Since $\mathbf{k} = \underline{\nabla}\theta$ is normal to these surfaces, $k^\mu = dx^\mu/dl$, where x^μ are the coordinates of a point in the ray, and l is an affine parameter. This means that geodesics satisfy $k^\mu k_\mu = \dfrac{dx^\mu}{dl}\dfrac{dx_\mu}{dl} = 0 \implies ds^2 = 0$, that is, the proper time along a geodesic is zero.

Affine parameter: it satisfies $\mathbf{k} = d/dl$. Therefore, $k^0 = dx^0/dl \implies \int dl = \int dx^0/k^0$, and so, the affine parameter is given by $l = t/k^0 + \text{const} = t/\omega + \text{const}$. [We put const = 0 in the thesis]. t is the proper time but as measured by a local Lorentz observer who sees angular frequency ω (because those frame-dependent quantities are being divided, the quotient, in this case the affine parameter, is frame independent).

Doppler effect: by the chain rule, $\omega_{\text{local}} = -\dfrac{\partial\theta}{\partial t_{\text{local}}} = -\dfrac{\partial\theta}{\partial x^{\alpha}_{\text{local}}}\dfrac{\partial x^{\alpha}_{\text{local}}}{\partial t_{\text{local}}} = -k_{\alpha}u^{\alpha}$, where u^{α} is the four-velocity of the observer relative to the source. For two observers, $\dfrac{\omega_1}{\omega_2} = \dfrac{k_{\alpha,1}u_1^{\alpha}}{k_{\alpha,2}u_2^{\alpha}}$.

Gravitational redshift: for a static observer $u^i = 0$, and then, $\omega = -k_0u^0$, but $k_0 = g_{0i}k^i + g_{00}k^0$, which implies $\omega = -(g_{00}k^0 + g_{0i}k^i)u^0$. If we choose the affine parameter of the observer as its proper time, $u^0 = dx^0_{\text{obs}}/d\tau$, we can normalize it to $u_0u^0 = -1 \implies g_{00}u^0u^0 = -1 \implies u^0 = \sqrt{\dfrac{-1}{g_{00}}}$ [it is -1 because of the signature +2, with Minkowski, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, but for a static observer, $dx = dy = dz = 0$, $t = \tau$, implies $ds^2 = -d\tau^2$ and $\dfrac{dx^{\mu}}{d\tau}\dfrac{dx_{\nu}}{d\tau} = -1$]

Photon reinterpretation: we make $\mathbf{p} = \hbar\mathbf{k}$, the momentum of the photon, and then, $\omega = -\mathbf{k}\cdot\mathbf{u} = -\frac{1}{\hbar}\mathbf{p}\cdot\mathbf{u}$, and we get the energy of a photon: $E = \hbar\omega = -\mathbf{p}\cdot\mathbf{u}$. It is also possible to put the amplitude of the wave in terms of the number density of the photons.

Static stars

General solution: the general solution of the Einstein field equations, assuming spherical symmetry is $ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

The function Λ: Inserting the metric into the Einstein field equation for the local time $E_{\hat{t}\hat{t}} = 8\pi T_{\hat{t}\hat{t}}$ $\implies e^{-2\Lambda} = 1 - 2m/r$, where $m = \int 4\pi\rho r^2 dr$.

The function Φ: Inserting the metric into the Einstein field equation for the local radius $E_{\hat{r}\hat{r}} = 8\pi T_{\hat{r}\hat{r}}$ $\implies \dfrac{d\Phi}{dr} = \dfrac{m + 4\pi r^3 p}{r(r-2m)}$ (it reduces to $d\Phi/dr = m/r^2$ in the Newtonian limit).

Hydrostatic equilibrium: $T^{\hat{r}\hat{\mu}}{}_{;\mu} = 0 \implies \dfrac{dp}{dr} + (p+\rho)\dfrac{d\Phi}{dr} = 0$

Tolman-Oppenheimer-Volkoff (TOV) equation: inserting the equation for Φ into the hydrostatic equilibrium condition: $\dfrac{dp}{dr} = -(\rho + p)\dfrac{m + 4\pi r^3 p}{r(r-2m)}$

External field: Schwarzschild solution. It is obtained by making $m = M$ for $r > R$, $p = 0$, and comparison with the Newtonian potential: $ds^2 = -\left(1 - \dfrac{2M}{r}\right)dt^2 + \dfrac{dr^2}{1-(2M/r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

Internal Schwarzschild field: for a constant-density spherical star, $m = \dfrac{4}{3}\pi\rho r^3$, the TOV equation is separable and yields $p = \dfrac{\sqrt{a^2-r^2} - \sqrt{a^2-R^2}}{3\sqrt{a^2-R^2} - \sqrt{a^2-r^2}}\rho$, where $a^2 = \dfrac{3}{8\pi\rho}$; which in turn gives us $e^{2\Phi} = \left(\dfrac{3}{2}\sqrt{1-\dfrac{2M}{R}} - \dfrac{1}{2}\sqrt{1-\dfrac{2M}{R^3}r^2}\right)^2$, where R is the radius of the star, with the condition that $e^{2\Phi(r=R)} = 1 - 2M/R$.

Stellar model: with the equation of state $\rho(p)$, and the density or pressure at the center $p(r = 0)$, the TOV equation can be integrated (numerically), giving the value of M and R (when the pressure goes to zero). Current efforts for neutron stars are in order to invert the process: given the values of M and R, "guess" the equation of state using different assumptions.

Radiation

From "Notes on Astrophysics": Intensity: $I_\nu = \dfrac{d\mathscr{E}_\nu}{dt\,\Delta A\,d\nu\,d\Omega}$. For blackbody radiation, $I_\nu = B_\nu(T)$. Flux: $F_\nu = \int I_\nu \cos\theta\, d\Omega$. Total flux: $F = \int F_\nu d\nu$.

Number density of photons in phase space $= \dfrac{\delta N}{V_{config} V_{phase}}$. Since it is a countable number, it is an invariant. $V_{\text{config}} = \Delta A\ dt$ ("cylinder" of particles that go through ΔA, geom. units). $V_{\text{phase}} = p^2 dp\, d\Omega$ (spherical coordinates). For a photon, in SI units, $\mathscr{E} = pc = h\nu = p^0 \implies dp = h\,d\nu/c$. Then, in geometrized units, $V_{\text{phase}} = h^3\nu^2 d\nu d\Omega \implies$ num. dens. $= \dfrac{\delta N}{\Delta A\ dt\, h^3\nu^2\, d\nu\, d\Omega}$.

Intensity invariant: we replace in the intensity equation $d\mathscr{E}_\nu = h\nu\delta N$, and then, $I_\nu = \dfrac{h\nu\delta N}{dt\Delta A d\nu d\Omega}$. Substituting this equation in the number density in phase space, num. dens. $= \dfrac{I_\nu}{h^4\nu^3}$. This means that $\dfrac{I_{\nu,\text{obs}}}{\nu^3_{\text{obs}}} = \dfrac{I_{\nu,\text{em}}}{\nu^3_{\text{em}}} \implies I_{\nu,\text{obs}} = I_{\nu,\text{em}}\left(\dfrac{\nu_{\text{obs}}}{\nu_{\text{em}}}\right)^3$ (affected by gravitational redshift).

Wien's displacement law: $\dfrac{T_{\text{obs}}}{\nu_{\text{obs}}} = \dfrac{T_{\text{em}}}{\nu_{\text{em}}} \implies T_{\text{obs}} = T_{\text{em}}\dfrac{\nu_{\text{obs}}}{\nu_{\text{em}}}$ (affected by gravitational redshift).

References

- H. Stephani, D. Kramer, M. MacCalum, C. Hoenselaers, E. Herlt. *Exact Solutions of Einstein's Field Equations*. Second Edition. Cambridge University Press, 2003.
- C. Misne, K. Thorne, J. Wheeler. *Gravitation*. Freeman and Company, 1970.
- A. Choudhuri. *Astrophysics for Physicists.* Cambridge University Press, 2010.
- G. Ludyk. *Einstein in Matrix Form.* Springer, 2013; § 1.8.
- Ø. Grøn. *Lecture Notes on the General Theory of Relativity.* Springer, 2009. § 10.3