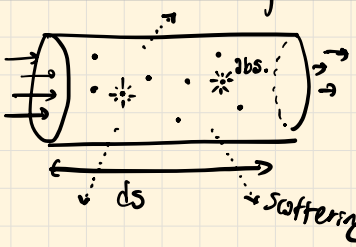
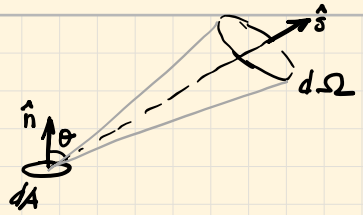
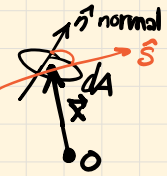
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From Bodenheimer, Laughlin Różyczka & Yorke "Numerical Methods in Astrophysics"

dE_ν : energy photons  within $d\Omega$ with freq. $[\nu, \nu+d\nu]$

$\Rightarrow$ intensity I_ν

$$dE_\nu = I_\nu(\vec{x}, \hat{s}, \nu, t)\, \underbrace{\hat{n}\cdot\hat{s}\, dA}_{dA\cos\theta}\, d\Omega\, dt$$

photon distr. funct. $f_R(\vec{x}, \vec{q}, t)$ ($\vec{q}$: momenta)

$\Rightarrow$ #photons $= f_R(\vec{x}, \vec{q}, t)\, d^3\vec{x}\, d^3\vec{q}$

- $d^3\vec{x}$: vol. where photons pass in a dt $= c\,dt\,\hat{n}\cdot\hat{s}\,dA$
- $d^3\vec{q}$: vol. in mom. space in sph. coord $q^2\,dq\,d\Omega$ (↳ mom. of a photon $q = h\nu/c$) $\Rightarrow (h\nu/c)(h/c)\,d\nu\,d\Omega$

then, the energy is (# photons)(energy per photon, $h\nu$)

$$dE_\nu = h\nu\, f_R(\vec{x}, \vec{q}, t)\, d^3\vec{x}\, d^3\vec{q}$$

$$= (h\nu/c)^2 (h^2\nu/c)\, d\nu\, d\Omega \cdot c\cdot dt\, (\hat{n}\cdot\hat{s})\, dA\, f_R$$

$$\Leftrightarrow I_\nu = \underbrace{\frac{h^4\nu^3}{c^2}}_{\equiv C_1} f_R$$

$\Rightarrow$ Boltzman (conserv. eqn.) for photons:

$$\frac{1}{C_1}\left[\frac{\partial I_\nu}{\partial t} + \underbrace{c(\hat{s}\cdot\vec{\nabla})}_{(\text{"vel."})} I_\nu\right] = \underbrace{\frac{1}{C_1}\left[\frac{\partial I_\nu}{\partial t}\right]}_{\text{sources}}$$

$$\longrightarrow \boxed{\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\hat{s}\cdot\vec{\nabla}) I_\nu = -\underbrace{\epsilon_\nu \rho\, I_\nu}_{\text{absorption}} + \underbrace{j_\nu \rho}_{\text{emission}}}$$

Radiative transfer eqn.

Tema/Topic: Moments of the intensity

For what follows, we define s_i (components of $\hat{s}$) as the direction cosines. Note that the magnitude of $\hat{s}$ is 1 but its components are not 1.

Moment zero: $\int_{4\pi} (\dots)\, d\Omega$

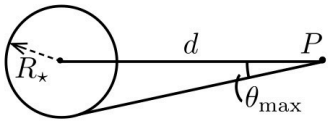
- mean intensity

$$J_\nu = \frac{1}{4\pi}\int_{4\pi} I_\nu\, d\Omega \quad : \text{interpretation:} \quad \frac{4\pi J_\nu}{c} = u_\nu \quad : \text{radiation energy density (energy/volume)}$$

First moment: $\int_{4\pi} (\dots)\, s_i\, d\Omega$

- $H^i_\nu \equiv \frac{1}{4\pi}\int_{4\pi} I_\nu\, s_i\, d\Omega$: interpretation $\equiv \frac{F^i_\nu}{4\pi}$ ← Flux (of photons)

Ex.: sph. coord., $F^{(r)}_\nu = \int_0^{2\pi}\int_0^{\pi} I_\nu \underbrace{\cos\theta}_{s_i} \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$



Uniformly radiating sphere: consider a sphere of radius $R_\star$ that radiates with uniform intensity I_0. z-axis along d. The flux that an observer located at P receives is

$$F = \int I\cos\theta\, d\Omega = I_0\int_0^{2\pi} d\phi \int_0^{\theta_{\max}} \cos\theta\sin\theta\, d\theta = I_0\cdot\frac{1}{2}\sin^2\theta_{\max}\cdot 2\pi, \text{ but}$$

since $\sin\theta_{\max} = R_\star/d$, $\implies F = I_0\pi(R_\star/d)^2$. Observe that the term $\pi(R_\star/d)^2$ is the solid angle subtended by the sphere at P (both making the star bigger or closer have the same effect).

The flux is, in general, a vector.

Second moment $\int_{4\pi} (\dots)\, s_i\, s_j\, d\Omega$

- $K^{ij}_\nu \equiv \frac{1}{4\pi}\int_{4\pi} I_\nu\, s_i\, s_j\, d\Omega$: interpretation $\frac{4\pi}{c}K^{ij}_\nu = P^{ij}_\nu$

radiation pressure

Tema/Topic: Moments of the RTE

For this part, we define the "source function" $S_\nu = j_\nu/\epsilon_\nu$ such that we replace the r.h.s. of the RTE with $-\epsilon_\nu \rho (I_\nu - S_\nu)$.

Zeroth moment $\frac{1}{4\pi}\int_{4\pi}(\cdots)\,d\Omega$ everywhere

$$\frac{1}{c}\frac{\partial J_\nu}{\partial t} + \vec{\nabla}\cdot\vec{H}_\nu = -\frac{1}{4\pi}\int_{4\pi}\epsilon_\nu \rho\,(I_\nu - S_\nu)\,d\Omega$$

First moment $\frac{1}{4\pi}\int_{4\pi}(\cdots)\,s_i\,d\Omega$ everywhere

$$\frac{1}{c}\frac{\partial H_\nu^i}{\partial t} + \sum_j \frac{\partial K^{ij}}{\partial x_j} = -\frac{1}{4\pi}\int_{4\pi} s_i\,\epsilon_\nu \rho (I_\nu - S_\nu)\,d\Omega$$

For thermal continuum emission & absorption, $S_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T)-1}$ black body radiation

Then,

$$\begin{cases} \frac{1}{c}\frac{\partial J_\nu}{\partial t} + \vec{\nabla}\cdot\vec{H}_\nu + \kappa_\nu \rho (J_\nu - B_\nu) = 0 \\ \frac{1}{c}\frac{\partial H_\nu^i}{\partial t} + \sum_j \frac{\partial K^{ij}}{\partial x_j} + \epsilon_\nu \rho H_\nu^i = 0 \end{cases}$$

(κ_ν: opacity)

Also written as

$$\frac{\partial U_\nu}{\partial t} + \vec{\nabla}\cdot\vec{F}_\nu - 4\pi\kappa_\nu \rho B_\nu + c\kappa_\nu \rho U_\nu = 0$$

Flux-limited diffusion

We express the flux as $\vec{F} = -\frac{c\lambda}{\kappa_R \rho}\vec{\nabla}U$ (λ: "flux limiter" = $\frac{1}{3}$ in the diffusion limit)

We substitute this in the zeroth moment equation and get

$$\frac{\partial U_\nu}{\partial t} - \underbrace{\frac{c}{3\kappa_R \rho}\nabla^2 U_\nu}_{\text{diffusion eqn.}} = \underbrace{4\pi\kappa_\nu \rho B_\nu - c\kappa_\nu \rho U_\nu}_{\text{sources (emission - absorption)}}$$

Discretization of the diffusion equation

The diffusion equation is

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \tag{1}$$

For the Flux-limited diffusion (FLD) approximation to radiation transport, $K = \frac{c}{3\kappa_R \rho}$ (and we would be missing the source terms).

Using finite differences where we abuse notation and use dt instead of Δt and no subindices,

$$\frac{\partial u}{\partial t} \approx \frac{u(t+dt,x) - u(t,x)}{dt} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(t,x+dx) - 2u(t,x) + u(t,x-dx)}{dx^2} \tag{3}$$

For this last derivative, we use the result from Bodenheimer et al ("Numerical methods in Astrophysics"), which implies that an evaluation at t will lead to an unconditionally unstable solution. Then we instead evaluate at $t \to t+dt$:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(t+dt,x+dx) - 2u(t+dt,x) + u(t+dt,x-dx)}{dx^2} \tag{4}$$

So that the diffusion equation is then discretized as

$$u(t+dt,x) = u(t,x) + \frac{Kdt}{dx^2}[u(t+dt,x+dx) - 2u(t+dt,x) + u(t+dt,x-dx)] \tag{5}$$

Defining $b \equiv Kdt/dx^2$, and sending all elements that depend on $t+dt$ to the l.h.s.,

$$bu(t+dt,x+dx) - (2b+1)u(t+dt,x) + bu(t+dt,x-dx) = -u(t,x) \tag{6}$$

This will define a *system* of equations on $t+dt$, knowing the solution at a previous timestep t. Because we have to solve this system of equations to find $u(t+dt,:)$, we call this an **implicit** solution method, as opposed to an *explicit* method, which is able to give us the solution $u(t+dt,:) =$ something that only depends on $u(t,:)$.

We see that for neighboring cells we can write the equation as

$$[\cdots \quad b \quad -2b-1 \quad b \cdots] \begin{bmatrix} \vdots \\ u(x-dx) \\ u(x) \\ u(x+dx) \\ \vdots \end{bmatrix}^{(t+dt)} = -\begin{bmatrix} \vdots \\ \vdots \\ u(x) \\ \vdots \\ \vdots \end{bmatrix}^{(t)} \tag{7}$$

so that we can write in matrix form

$$\begin{bmatrix} -2b-1 & b & 0 & 0 & \cdots & \cdots \\ b & -2b-1 & b & 0 & \cdots & \cdots \\ 0 & b & -2b-1 & b & \cdots & \cdots \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & b & -2b-1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{nx} \end{bmatrix}^{(t+dt)} = - \begin{bmatrix} u_0 + bu_L \\ u_1 \\ u_2 \\ \vdots \\ u_{nx} + bu_R \end{bmatrix}^{(t)} \tag{8}$$

where u_L is the boundary condition to the left of the cell 0 in x, and u_R is the boundary condition to the right of the last cell (number nx) in x.