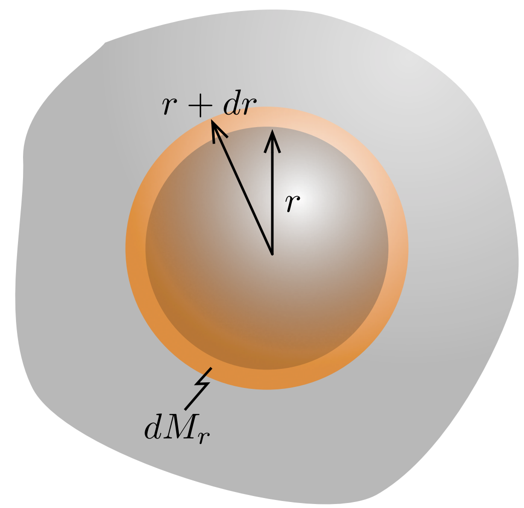
Stellar structure and evolution

Contents

Stellar structure equations

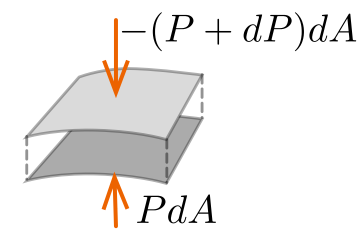
I. Conservation of mass:



The matter enclosed in a spherical shell of thickness dr has a mass $dM_r = 4\pi r^2 \rho dr$. The first stellar structure equation is $\dfrac{dM_r}{dr} = 4\pi r^2 \rho$.

II. Hydrostatic equilibrium: net force caused by the gradient of pressure: $-dPdA$. $\sum F_r = 0 \implies -dPdA - \dfrac{GM_r}{r^2}\rho dr dA = 0$.

The second stellar structure equation is $\dfrac{dP}{dr} = -\dfrac{GM_r}{r^2}\rho$.

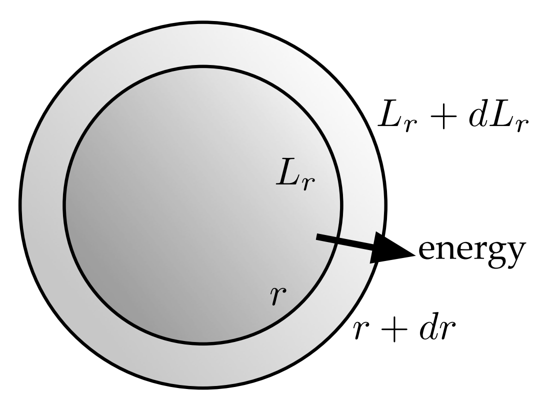


[If the density were constant (obviously not the case), the first structure equation yields $M_r = Mr^3/R^3$, and the second equation, $dP/dr = GMr^3/(r^2R^3)\rho \implies P = GM\rho \int rdr/R^3 = 3GM^2/(8\pi R^4)$]

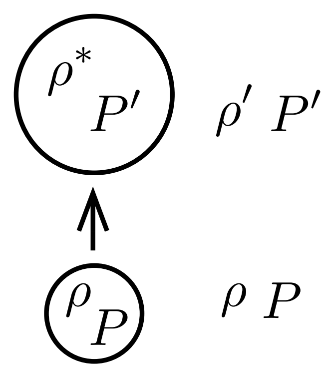
Equation of state: inside of the star, the ideal gas equation of state holds, $PV = nRT$; but $M = \mu n$ (the mass is the mass of one mol times the number of moles) $\implies P = \dfrac{\rho}{\mu}RT$

III. Energy transport due to nuclear reactions: consider two spherical shells like in the figure, inside a star. Let L_r be the total energy flux per unit time that flows outwards across the shell of radius r. Then, dL_r is the energy input to the energy flux made by the region $[r, r+dr]$. If ε is the rate of energy generation per unit mass per unit time (by nuclear reactions), then,

$dL_r = 4\pi r^2 dr \cdot \rho \cdot \varepsilon \implies \dfrac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$ (third structure equation).



IV-a. Energy transport in radiative zones: $U_{\text{rad}} = aT^4 \implies \dfrac{dU_{\text{rad}}}{dr} = 4aT^3\dfrac{dT}{dr}$. In a radiative zone, the mean free path of radiation is $\lambda = 1/(\kappa\rho)$ [taking Thomson's electron scattering opacity] $\approx 2\,\text{cm} \ll R_\odot$. That means that radiation behaves like diffusion, and therefore, we can use the diffusion approximation: $F_{\text{rad}} = -D\nabla_r U_{\text{rad}}$, with the diffusion coefficient $D = c/(3\kappa\rho)$. Therefore, $F_{\text{rad},r} = -\dfrac{4ac}{3}\dfrac{T^3}{\kappa\rho}\dfrac{dT}{dr}$. Using the luminosity ($L_r = 4\pi r^2 F_{\text{rad},r}$), we get the energy transfer equation for the radiative regime, which is not valid near the atmosphere (→ § Stellar atmospheres: radiation transport): $\dfrac{dT}{dr} = -\dfrac{3}{16\pi ac}\dfrac{\kappa\rho L}{r^2T^3}$.



Convection inside stars: a blob of material of density ρ, pressure P is in hydrostatic equilibrium with its surroundings, of same density and pressure. The blob is transported adiabatically to another place, where the surrounding density and pressure are ρ', P', respectively. The blob expands/contracts so it has the same pressure than the surroundings, changing its density to ρ^*. If $\rho^* < \rho'$, the blob will be buoyant and continue to move upwards (→ convection), but if $\rho^* < \rho'$, it will return to the original position.

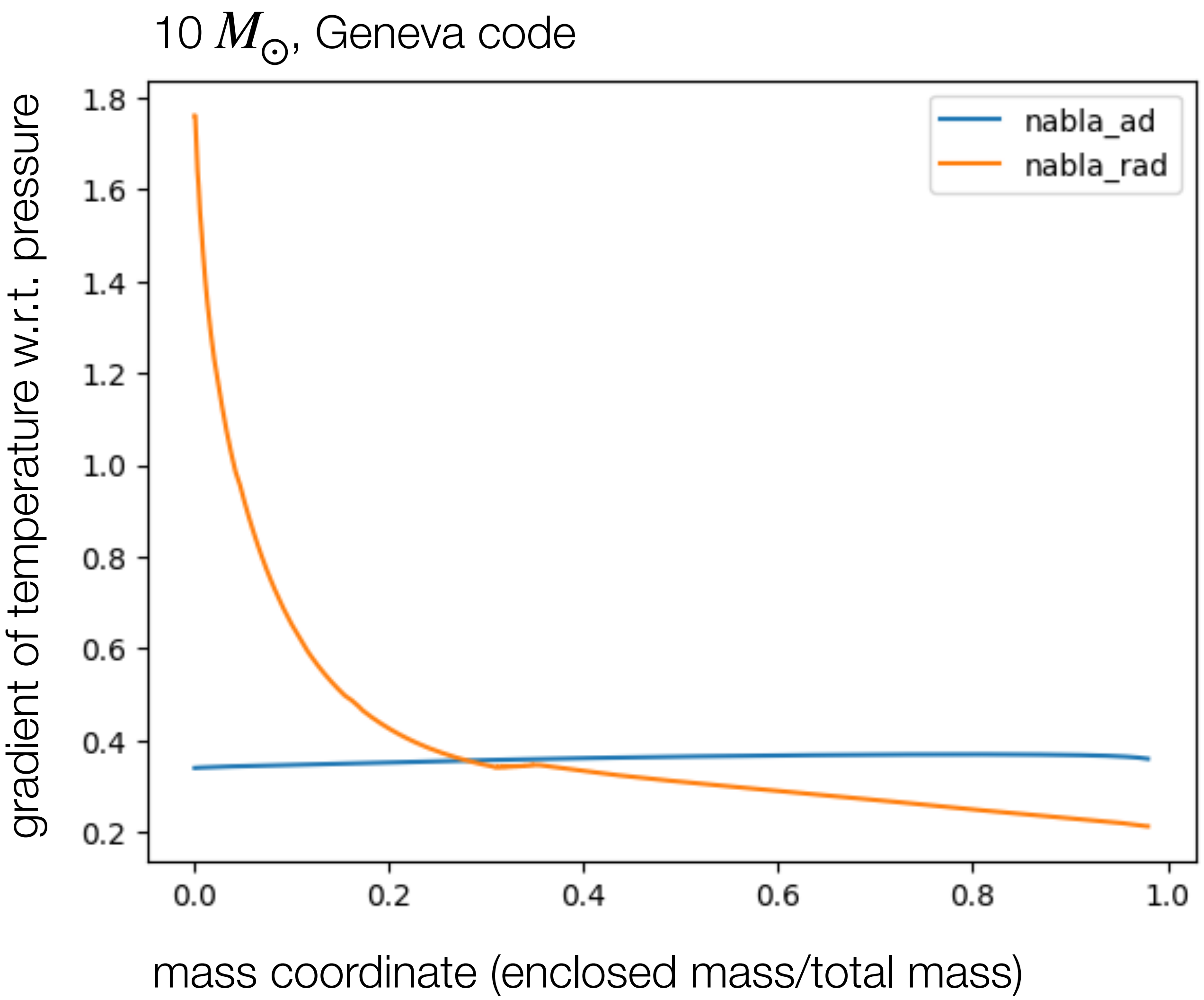
IV-b. Energy transport in convective zones: Since the convection process is adiabatic, $\rho^* = \rho(P'/P)^{1/\gamma}$. Now, $P' = P + \dfrac{dP}{dr}\Delta r$

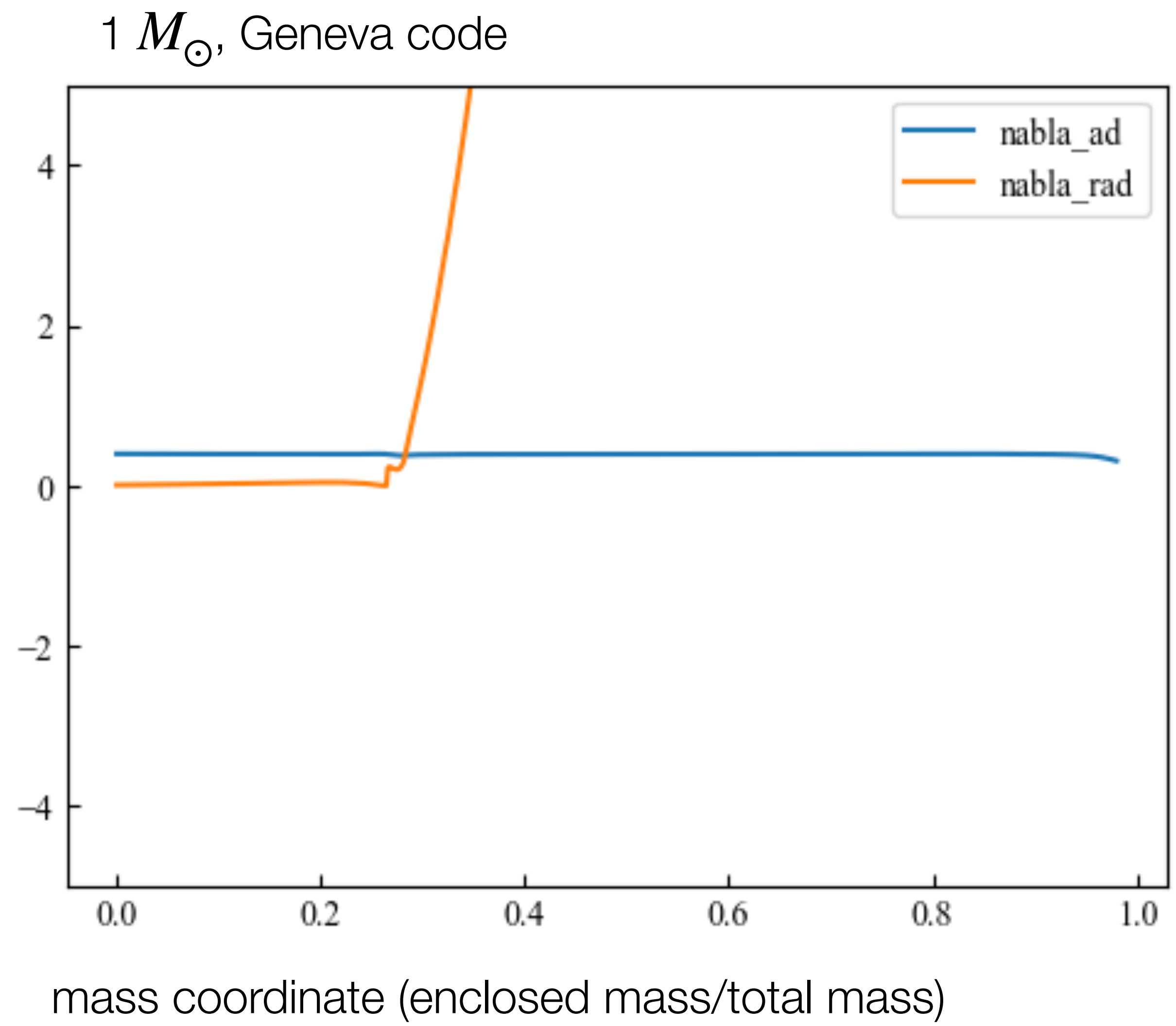
$\implies \rho^* = \rho\left(1 + \dfrac{1}{P}\dfrac{dP}{dr}\Delta r\right)^{1/\gamma}$. [Binomial expansion]

$\implies \rho^* = \rho + \dfrac{\rho}{\gamma P}\dfrac{dP}{dr}\Delta r$. We can describe the change in the surrounding density field, as well, as $\rho' = \rho + \dfrac{d\rho}{dr}\Delta r$. Using the ideal equation of state $[\rho = P/(RT)]$, we get $\rho' = \rho + \dfrac{1}{RT}\dfrac{dP}{dr}\Delta r - \dfrac{P}{T^2}\dfrac{dT}{dr}\Delta r$

$\implies \rho' = \rho + \dfrac{\rho}{P}\dfrac{dP}{dr}\Delta r - \dfrac{\rho}{T}\dfrac{dT}{dr}\Delta r$. Finally,

$\rho^* - \rho' = \left[-\left(1 - \dfrac{1}{\gamma}\right)\dfrac{\rho}{P}\dfrac{dP}{dr} + \dfrac{\rho}{T}\dfrac{dT}{dr}\right]\Delta r$, which means that the atmosphere is stable if $\left|\dfrac{dT}{dr}\right| < \left(1 - \dfrac{1}{\gamma}\right)\dfrac{T}{P}\left|\dfrac{dP}{dr}\right|$. If the temperature gradient is only slightly steeper than the critical gradient, the typical stellar energy fluxes are obtained. The equation for energy transport in convective zones, is, then, $\dfrac{dT}{dr} = \left(1 - \dfrac{1}{\gamma}\right)\dfrac{T}{P}\dfrac{dP}{dr}$.





Structure of high-mass vs a low-mass star

There is a parameter called $\nabla \equiv \ln T / \ln P$ which describes how temperature changes with pressure (confusing symbol, this is not a vector!). For an idealized radiative and convective regime, one can compute the value of this gradient as $\nabla_{\rm rad}$ and $\nabla_{\rm ad}$, respectively ("ad" means adiabatic because convection is good at trapping heat). The smallest of the two values determines how energy is transported inside the star. In the plots, we have the values of the nablas computed by the Geneva code for a high-mass (left) and a low-mass (right) star. One can easily see that the high-mass star has a convective core ($\nabla_{\rm ad} < \nabla_{\rm rad}$) and radiative envelope ($\nabla_{\rm rad} < \nabla_{\rm ad}$). Exactly the opposite is true in a star like the Sun.

Order-of-magnitude calculations

Mass-density: $\rho \approx \dfrac{3M}{4\pi R^3} \approx 0.24\dfrac{M}{R^3}$

Number density: $n \approx \dfrac{3M}{m_p 4\pi R^3} \approx 0.24\dfrac{M}{m_p R^3}$ (protons dominate mass)

Pressure: from the second structure equation, $P \approx \dfrac{3GM^2}{8\pi R^4} \approx 0.12\dfrac{GM^2}{R^4}$

Main-sequence star: made of ideal gas. Its temperature can be estimated as $T = \dfrac{P}{nk_B} \approx \dfrac{0.12GM^2R^{-4}}{0.24Mm_p^{-1}R^{-3}k_B}$, for the Sun, $T \approx 10^7$ K.

Mass-luminosity relation: order of magnitude version of the mass equation: $\dfrac{M}{R} \sim R^2\rho$ (1), hydrostatic equation: $\dfrac{P}{R} \sim \dfrac{GM\rho}{R^2}$ (2), energy transport equation for radiative zones: $\dfrac{T}{R} \sim \dfrac{K\rho L}{R^2T^3}$ (3) (K is the constant mass-specific absorption coefficient, or opacity), ideal gas $P \sim \rho T/\mu$ (4) (μ is the mean molecular mass, [mass]/mol). Let's substitute the density from (1) everywhere. Then, (3) becomes $T^4 \sim KLM/R^4 \implies L \sim T^4R^4/(KM)$. To substitute T: (4) becomes $P \sim MT/(\mu R^3) \implies T \sim P\mu R^3/M$, but with (2), $T \sim (GM^2/R^4)\,\mu R^3/M \sim G\mu M/R$. Then, we arrive at the mass-luminosity relation, $L \sim G^4\mu^4M^3/K$.

Proportions and stellar lifetime: then, $T \propto M/R$ and $L \propto M^3$. From black body radiation, we know that $L \propto R^2T^4$. Then, comparing luminosities, $M^3 \propto M^2T^2$ or $M \propto T^2$. Using this to substitute for mass in the mass-luminosity relation, $L \propto T^6$. A star is powered by nuclear energy, the amount of which is proportional to the mass. The luminosity of the star is its power output, so $L \propto M/\tau$. Then, the lifetime of a star is $\tau \propto M^{-2}$, which means that more massive stars live shorter lives.

Stellar evolution

Nuclear reactions

Proton-proton (PP) chain: (notation: $^{\text{nucleons}}_{\text{protons}}\text{Element}$) the reaction happens in several steps:

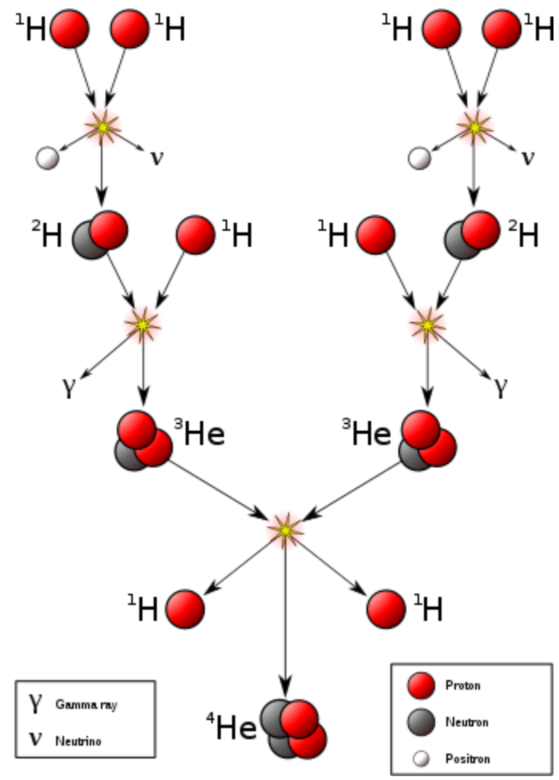
1) $^1_1\text{H} + ^1_1\text{H} \longrightarrow ^2_1\text{H} + e^+ + \nu_e$ (requires β^+ decay: $p + \text{energy} \rightarrow n + e^+ + \nu_e$)

2) $^2_1\text{H} + ^1_1\text{H} \longrightarrow ^3_2\text{He} + \gamma$

3) $^3_2\text{He} + ^3_2\text{He} \longrightarrow ^4_2\text{He} + 2^1_1\text{H}$

overall reaction:

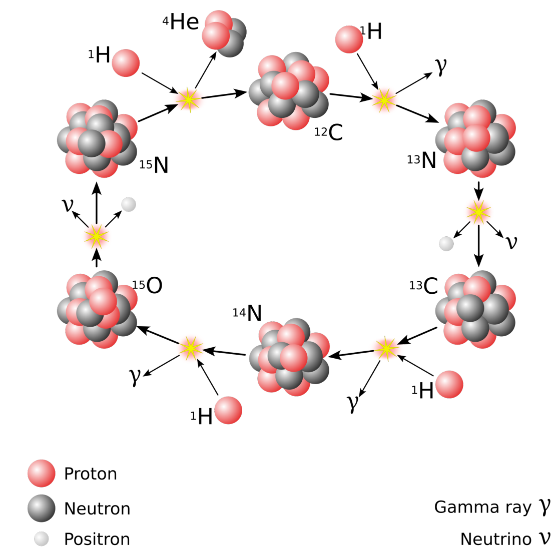
$4^1_1\text{H} \longrightarrow ^4_2\text{He} + 2e^+ + 2\nu_e + 2\gamma + \text{kinetic en.}$



Time that fusion can power a Sun-like star: Since luminosity is power, $t \sim \dfrac{E_{\text{tot}}}{L_\odot}$. In nuclear fusion of hydrogen to helium, $E_{\text{tot}} = N\Delta mc^2$, where $N = M_\odot/(4m_H)$ is the number of 4 hydrogen atoms in the star (4 are needed for the reaction to happen) and $\Delta m \approx 25.71\,\text{MeV}/c^2$ is the difference in mass between 4^1_1H and ^4_2He (it corresponds to the binding energy released in the nuclear reaction). The percentage of mass that transforms into energy is $\varepsilon = \Delta m/(4m_\text{H}) \approx 0.7\,\%$ (sometimes called efficiency). So, $E_{\text{tot}} \sim 0.007 M_\odot c^2 \sim 1.3 \cdot 10^{45}\,\text{J}$ $\implies t \sim \dfrac{1.3 \cdot 10^{45}\,\text{J}}{3.8 \cdot 10^{26}\,\text{J s}^{-1}} \sim 3.3 \cdot 10^{18}\,\text{s} \sim 10^{11}\,\text{yr}$. This assumes a fully convective star (all H available for burning); in reality it is only about 10% for a Sun-like star.

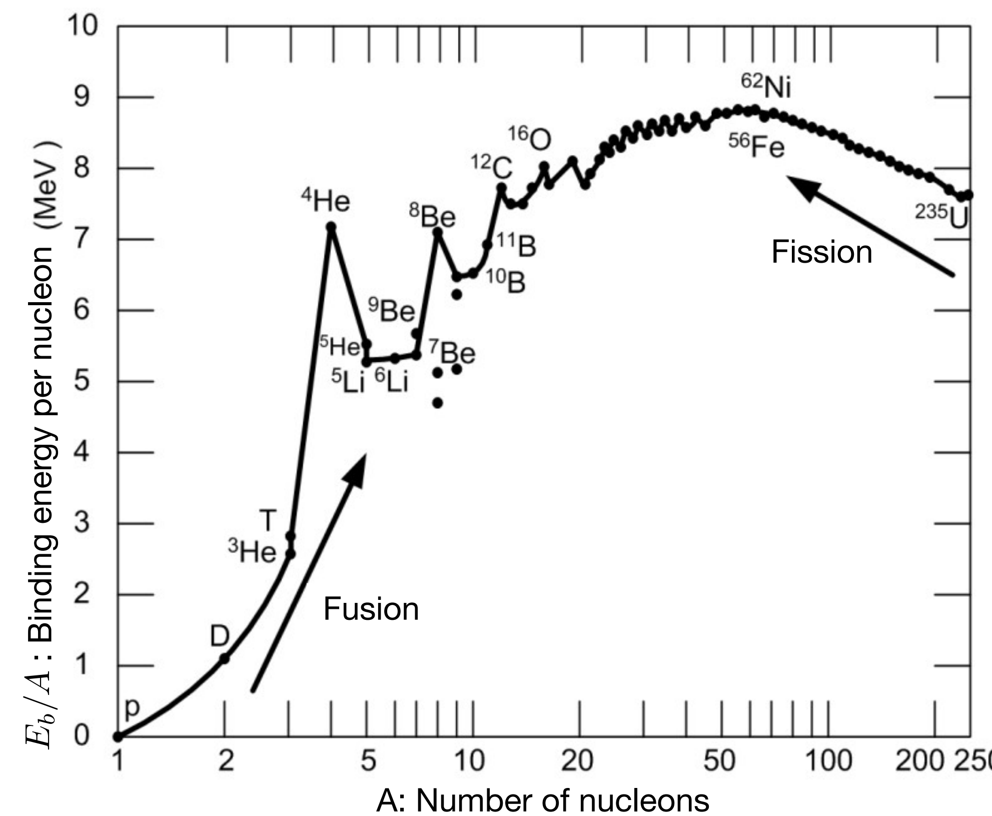
Nuclear potential: Two protons at distances much larger than the nuclear radius repel each other (Coulomb potential). However, at distances smaller than a nuclear radius, the potential cannot be the Coulomb one (repels nuclei), but it must be attractive: short-range nuclear forces overcome electrostatic forces. At $r = 0$, this potential should be $\sim -30\,\text{MeV}$ (from the mass deficiency), and at around the nuclear radius ($r \approx 1\,\text{fm}$), it should become the Coulomb potential, $\sim +2e^2/r$.

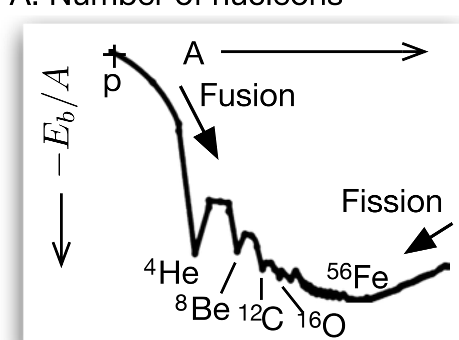
Temperature for fusion (quantum tunnelling): considering for a proton $p = h/\lambda$, and $r_{\text{action}} \rightarrow \lambda$, we get kinetic energy $= \dfrac{1}{2}\mu v^2 = \dfrac{p^2}{2\mu} = \dfrac{h^2}{2\mu\lambda^2}$, where $\mu = m_p^2/(2m_p) = m_p/2$ is the reduced mass (two body problem). This has to be equal to Coulomb barrier $\sim \dfrac{2e^2}{\lambda}$, so we solve for the wavelength $\lambda \sim \dfrac{h^2}{2e^2\mu}$. Then, by the equipartition theorem, $\dfrac{3}{2}k_B T = \dfrac{h^2}{2\mu\lambda^2} \implies T \sim 10^7\,\text{K}$, which is the same order of magnitude of the temperature of the Sun we estimated in § Order of magn. approx.



CNO cycle: another way to fusion hydrogen to helium, but, using carbon, nitrogen and oxygen as catalysts (i.e., the fusion of H with isotopes of N, C and O [+decays] releases helium at some point). It requires the previous presence of N, C and O.

Binding energy: mass-energy difference between isolated and bound components: $E_b = [\bar{Z}m_p + (\bar{A} - \bar{Z})m_n - m_{\text{nuc}}]c^2$, where $\bar{Z}$ is the number of protons, $\bar{A}$ is the number of nucleons and m_{nuc} is the experimental value of the mass of the bound nucleus. The values are plotted in the figure (seeing it reflected helps a lot!). For elements to the left of iron, fusion with a proton is energetically favorable and the processes are exothermic (except for ^{4}He, ^{12}C, ^{16}O, that are stable). For elements heavier than iron, energy has to be invested to create elements even heavier, and nuclear fission (radioactive decay) releases energy when heavy elements turn into lighter ones.





Alpha process/ladder: when H is exhausted from a star, He starts burning and fusing into heavier elements (alpha particle = He nucleus). This series of reactions creates heavier and heavier elements until Ni and Fe are reached.

(stellar evolution presentation)