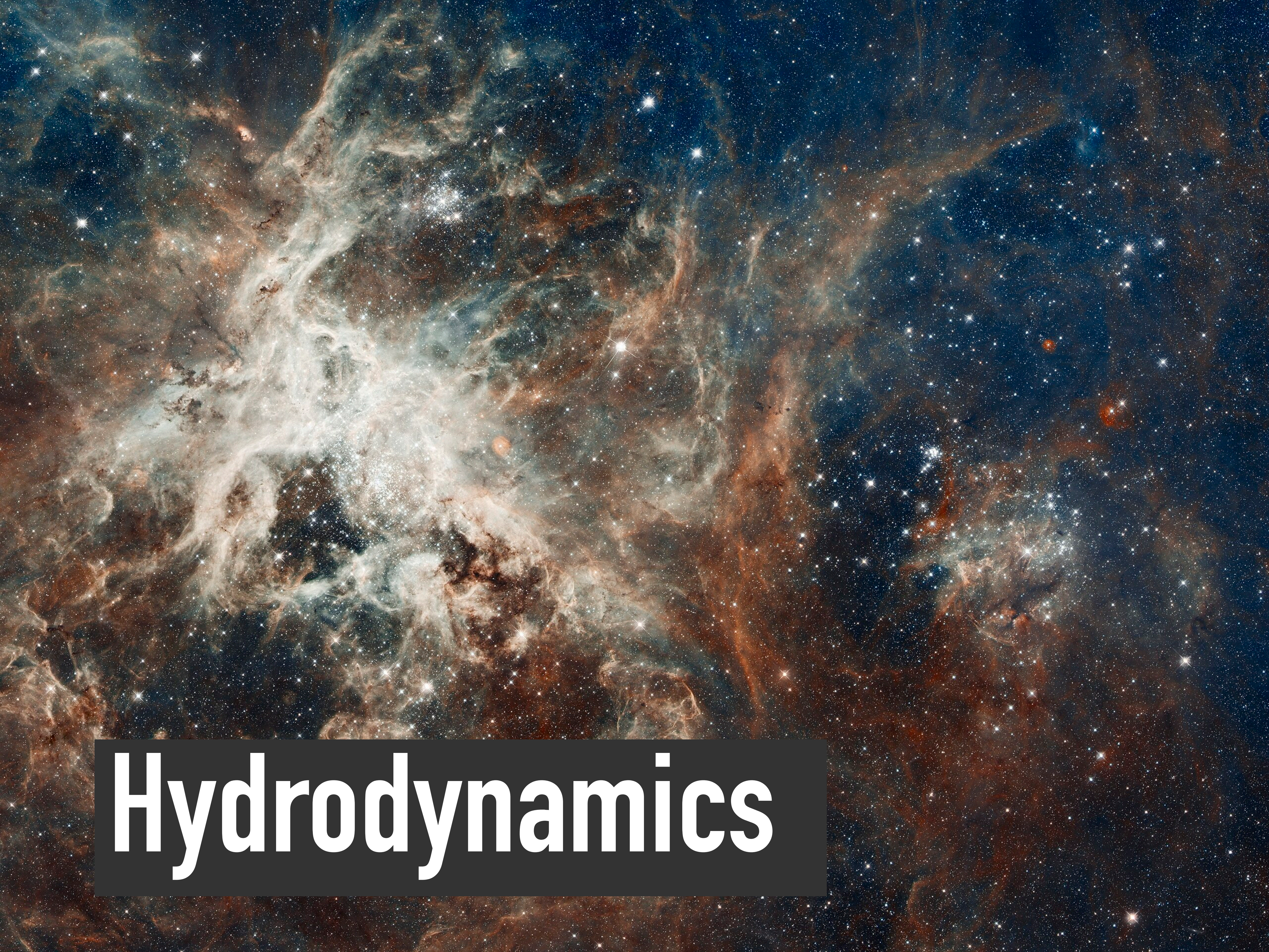


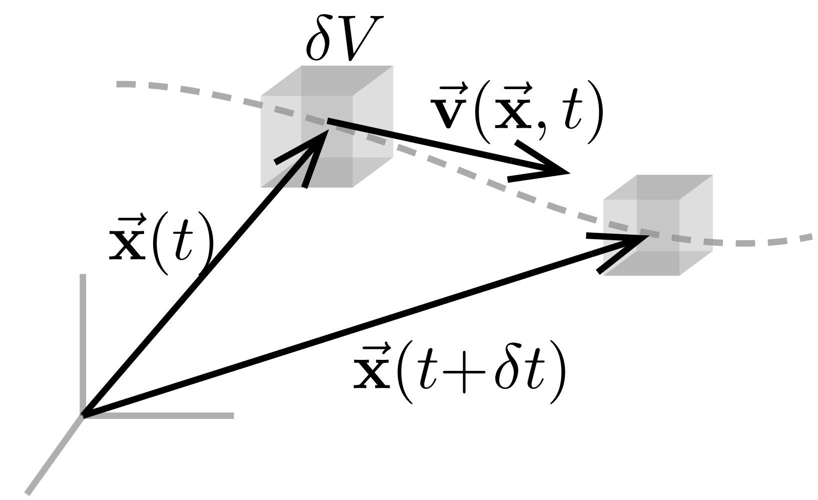
Hydrodynamics

Equations of motion

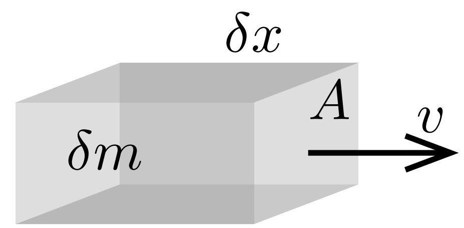
Lagrangian derivative and Eulerian derivative

The thermodynamic state of a small volume of fluid δV is given by its density, $\rho(\vec{\mathbf{x}}, t)$ and its temperature $T(\vec{\mathbf{x}}, t)$. Let $\vec{\mathbf{v}}(\vec{\mathbf{x}}, t)$ be the velocity of the small volume located in $\vec{\mathbf{x}}$ at time t.

Time derivatives: Let $\vec{\mathbf{x}}(t + \delta t) = \vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t$. For some quantity $Q(\vec{\mathbf{x}}, t)$, the time derivative is $\dfrac{dQ}{dt} = \lim_{\delta t \to 0} \dfrac{Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t, t + \delta t) - Q(\vec{\mathbf{x}}, t)}{\delta t}$. But making a Taylor expansion, $Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t, t + \delta t) = Q(\vec{\mathbf{x}}, t) + \delta t \dfrac{\partial Q}{\partial t} + \delta t \vec{\mathbf{v}} \cdot \vec{\nabla} Q$, which gives us $\dfrac{dQ}{dt} = \dfrac{\partial Q}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} Q$. The total derivative is called the Lagrangian derivative and the partial one, the Eulerian derivative.



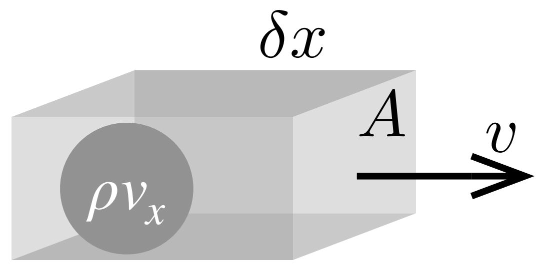
Continuity equation: for the change of mass, $\frac{\delta m}{\delta t} = \frac{\rho \delta V}{\delta t} = \frac{\rho A \delta x}{\delta t} = \rho A v$. In general, however, $\frac{\partial}{\partial t} \int \rho dV = - \oint \rho \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$. Using the Gauss's theorem, $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) = 0$.



Euler equation (momentum equation): Newton's second law: $\rho \delta V \frac{d\vec{\mathbf{v}}}{dt} = \delta \vec{\mathbf{F}}_{\text{body}} + \delta \vec{\mathbf{F}}_{\text{surface}}$. But $\delta \vec{\mathbf{F}}_{\text{body}} = \rho \delta V \vec{\mathbf{F}}$ (where $\mathscr{D}[\vec{\mathbf{F}}] = F/M$, i.e., specific force = acceleration) and $\delta \vec{\mathbf{F}}_{\text{surface}} = - \oint P d\vec{\mathbf{A}} = - \int \vec{\nabla} P dV$. Inserting everything and changing the Lagrangian derivative, $\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = - \frac{1}{\rho} \vec{\nabla} P + \vec{\mathbf{F}}$.

Advection vs convection: convection is the movement of a fluid mainly due to density gradients created by thermal gradients; advection is the more general transport of material or physical quantity by the velocity of the fluid.

Advection equation: the general equation $\frac{\partial \psi}{\partial t} + \vec{\nabla} \cdot (\psi \vec{\mathbf{v}}) = 0$ states the advection for a conserved quantity described by a scalar field ψ due to the transport in the velocity field $\vec{\mathbf{v}}$. If $\vec{\nabla} \cdot \vec{\mathbf{v}} = 0$ (incompressible flow / solenoidal) $\implies \frac{\partial \psi}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} \psi = 0$.



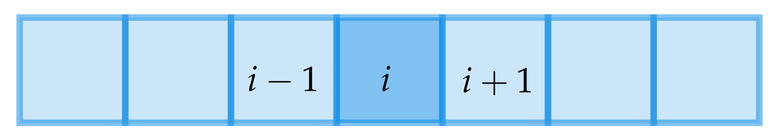
Conservative form of the momentum equation: consider a element of fluid that contains a momentum in the x direction $\delta \mathscr{P}_x = \rho v_x \delta V$. Then, let's consider how this momentum changes in time in a similar way to what we did for the continuity equation: $\frac{\delta \mathscr{P}_x}{\delta t} = \frac{\rho v_x \delta V}{\delta t} = \rho v_x A \frac{\delta x}{\delta t} = \rho v_x A v$. Here, v_x is the x-component of the velocity inside of the fluid element (used to compute the momentum of said fluid element), and v is the velocity across the wall, that is, the velocity that transports (advects) the momentum out of the fluid element across the wall A. They are the same velocity field, just measured at different positions (they reduce to the same vector value when $\delta V \to 0$). Then, in general for all the walls, $\frac{\partial}{\partial t}\int \rho v_x dV = -\oint \rho v_x \vec{\mathbf{v}} \cdot d\overrightarrow{\mathbf{A}}$ (this means in words that any momentum decrease inside of the fluid element in the x-direction must happen because it is lost through the walls of the element). Using Gauss's theorem, $\implies \frac{\partial}{\partial t}(\rho v_x) + \overrightarrow{\nabla} \cdot (\rho v_x \vec{\mathbf{v}}) = 0$ (momentum is advected and conserved). This is true if there are no forces. If there are forces, they act as the sources of the equation and we have $\frac{\partial}{\partial t}(\rho v_x) + \overrightarrow{\nabla} \cdot (\rho v_x \vec{\mathbf{v}}) = -\frac{\partial P}{\partial x} + \rho F_x$. One can show that this equation is equivalent to the Euler equation. There are similar equations for the y and z directions.

Conservation of energy: the same reasoning can be applied for the scalar field defined by the sum of the (volumetric) kinetic energy $\frac{1}{2}\rho v^2$ and the (volumetric) internal energy $\rho\epsilon$ and we obtain an equation of conservation of energy, where the sources can be: heating/cooling of the gas by thermal contraction/expansion, mechanical heating/cooling by external forces, heating/cooling by other processes such as radiation (absorption/emission), etc. In addition, we need an equation of state (relation between pressure and density, for example, the ideal gas law).

Discretization

For the numerical solution of the hydrodynamical equations, the basic idea is to divide the space and time in discrete steps; the division of space is called a *grid*. Once this division has been made, then, we approximate the differential operators with finite differences ($d \rightarrow \Delta$).

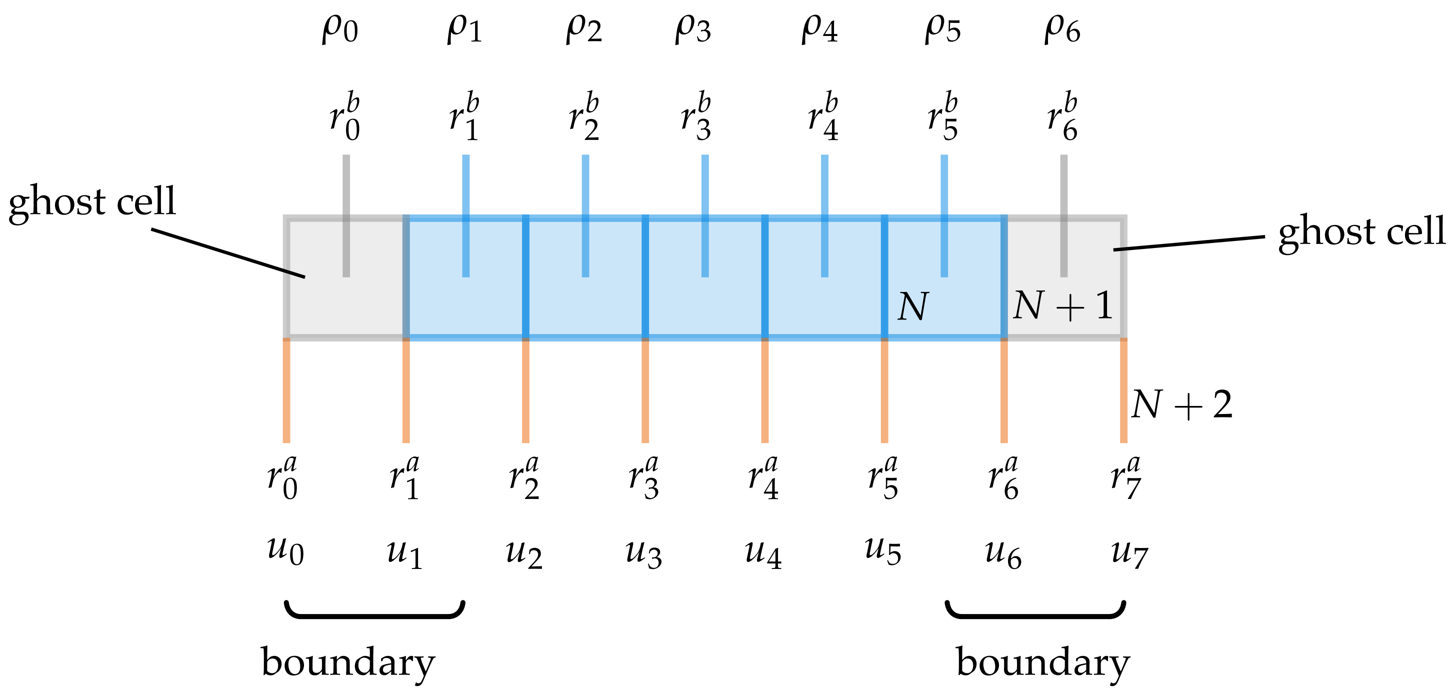
Not all discretizations lead to a stable approximation of the solution; a detailed analysis is required in many cases. For example, consider the approximation of a gradient. Which discretization is more convenient? The last one, that combines information in equal manner from the left and the right of the analyzed cell, seems to be well motivated. However, it can be shown that in an advection problem, like the one we are trying to solve, and if combined by a straightforward time approximation, this approximation leads to an unstable solution for any time.



$$\frac{\Delta f}{\Delta x} = \begin{cases} \dfrac{f_i - f_{i-1}}{x_i - x_{i-1}} \\[2ex] \dfrac{f_{i+1} - f_i}{x_{i+1} - x_i} \\[2ex] \dfrac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} \end{cases}$$

?

In the numerical solution of the problem, we use a *staggered* grid, where some quantities are defined at the cell centers, and others, at the cell walls. As it is shown in the figure, we define the grid r_i^a at the cell walls, and the grid r_i^b at the cell centers (the actual grid should have an appropriately high number of cells). The grid should extend from $r_1^a = 0$ to $r_N^a \approx 1.4 R_\star$, where $R_\star$ is the radius of the star determined in the first part, i.e., with the Lane-Emden equation. The cells we use here are in the shape of a spherical shell.



In this staggered grid, we define the scalar quantities, i.e., the density and the pressure, at the cell centers, and the vector quantities, that is, velocity, force and momentum, at the cell walls. In the figure, we see some cells in gray, labeled *ghost cells*. They are meant to facilitate the setup and application of the boundary conditions, but they do not form part of the solution of the problem.

Equations of hydrodynamics in 1D with spherical symmetry

conservation of mass $\Longrightarrow$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0.$$

conservation of momentum $\Longrightarrow$

$$\frac{\partial}{\partial t}(\rho u) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u u) = -\frac{\partial p}{\partial r}.$$

momentum density: $w := \rho u$

momentum flux: $F^w := wu$

equation of state:

$p = K\rho^\gamma$ ⟵ polytropic equation of state

speed of sound, if γ adiabatic:

$$c_s = \sqrt{\frac{\gamma p}{\rho}}.$$

Overview: upwind scheme with finite differences

The independent variables of the system of equations formed by the continuity equation and the momentum equation are the velocity u and the momentum ρ. The pressure can always be determined by the equation of state. The conservation form of the equations allows us to notice that the left hand side of both equations has the same form:

$$\frac{\partial}{\partial t} f + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F^f) = \text{sources},$$

where f is a quantity (here, density or velocity), and $F^f := fu$ is its flux. We split the task of updating the quantity f in time into two steps: an *advection step*, in which the left hand side of the equation is applied, and the *sources* or *forces step*, in which the external forces or sources are applied. The continuity equation, having no sources, only requires the advection step; the momentum equation, in the other hand, requires both steps. This process is called **operator splitting**. In each of the steps, the finite differences method is applied, and the fluxes must be carefully computed (the principle that we use for flux calculation is called the *first order upwind method.*

Time stepping

The selection of the time step Δt is not completely arbitrary, but it should satisfy the CFL-criterion (after Courant, Friedrich and Levy): the time step should be small enough such that no perturbation should propagate further than $C\Delta r$, where Δr is the width of a cell and C has a value lower than 0.75. The propagation speed of a perturbation in each cell is given by $c_s + |u|$, where c_s is the speed of sound in this case. Then,

$$\Delta t \leq C \min_i \left(\frac{\Delta r}{c_{s_i} + |u_i|} \right),$$

where the operator $\min_i$ returns the minimum value of the quantity among all the grid cells.