Magnetohydrodynamics

MHD equations

Curl of a cross product:
$\vec{\nabla}\times(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = (\vec{\mathbf{B}}\cdot\vec{\nabla})\vec{\mathbf{A}} - \vec{\mathbf{B}}(\vec{\nabla}\cdot\vec{\mathbf{A}}) + (\vec{\nabla}\cdot\vec{\mathbf{B}})\vec{\mathbf{A}} - (\vec{\mathbf{A}}\cdot\vec{\nabla})\vec{\mathbf{B}}$
. First, we apply the product rule (marking which vector $\vec{\nabla}$ is acting on) and then we expand using the BAC-CAB rule.

Gradient of a dot product:
$\vec{\nabla}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) = \dot{\vec{\nabla}}(\dot{\vec{\mathbf{A}}}\cdot\vec{\mathbf{B}}) + \dot{\vec{\nabla}}(\vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{B}}})$ (the dot marks where the derivative acts on). But applying the BAC-CAB rule to the cross product of a curl,
$\vec{\mathbf{A}}\times(\vec{\nabla}\times\vec{\mathbf{B}}) = \dot{\vec{\nabla}}(\vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{B}}}) - (\vec{\mathbf{A}}\cdot\vec{\nabla})\vec{\mathbf{B}}$, one of the terms appear. Making $\vec{\mathbf{A}} \leftrightarrows \vec{\mathbf{B}}$ and substituting, we find
$\vec{\nabla}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) = (\vec{\mathbf{A}}\cdot\vec{\nabla})\vec{\mathbf{B}} + (\vec{\mathbf{B}}\cdot\vec{\nabla})\vec{\mathbf{A}} + \vec{\mathbf{A}}\times(\vec{\nabla}\times\vec{\mathbf{B}}) + \vec{\mathbf{B}}\times(\vec{\nabla}\times\vec{\mathbf{A}})$
.

Ohm's law: $\vec{\mathbf{j}} = \sigma\vec{\mathbf{E}}$ or $\eta\vec{\mathbf{j}} = \vec{\mathbf{E}}$, but since the magnetic field also contributes to the force on charges, $\vec{\mathbf{j}} = \sigma(\vec{\mathbf{E}} + \vec{\mathbf{v}}\times\vec{\mathbf{B}})$.

Ampère-Maxwell equation: for astrophysical plasmas with $v \ll c$, we ignore the displacement current term. So, $\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{j}}$.

Electric field: combining Ampère's and Ohm's laws, $\vec{\mathbf{E}} = \dfrac{\vec{\nabla}\times\vec{\mathbf{B}}}{\mu_0\sigma} - \vec{\mathbf{v}}\times\vec{\mathbf{B}}$. For astrophysical plasmas, $\vec{\mathbf{B}}$ is more relevant than $\vec{\mathbf{E}}$, because charges are well mixed.

Induction equation: Faraday's law:
$\dfrac{\partial\vec{\mathbf{B}}}{\partial t} = -\vec{\nabla}\times\vec{\mathbf{E}}$. Substituting the electric field,
$\dfrac{\partial\vec{\mathbf{B}}}{\partial t} = \vec{\nabla}\times(\vec{\mathbf{v}}\times\vec{\mathbf{B}}) + \eta\nabla^2\vec{\mathbf{B}}$, where $\eta = 1/(\mu_0\sigma)$.
[Application of the "BAC-CAB" rule for gradients, one of the terms will be $\vec{\nabla}\cdot\vec{\mathbf{B}} = 0$. $\vec{\nabla}\times(\vec{\mathbf{v}}\times\vec{\mathbf{B}})$ is called the advection term.]

Euler equation: modification: $\vec{\mathbf{F}} \to \vec{\mathbf{F}} + \dfrac{1}{\rho}\vec{\mathbf{j}}\times\vec{\mathbf{B}}$.
Substituting $\vec{\mathbf{j}}$ (with Ampère's law), using the gradient of a dot product when both vectors are equal,

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}}\cdot\vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{F}} - \frac{1}{\rho}\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{\vec{\mathbf{B}}\cdot\vec{\nabla}}{\mu_0\rho}\vec{\mathbf{B}}.$$

The first new term is the magnetic field pressure and the second term, the magnetic tension force, which tends to straighten magnetic field lines.

Summary: for an astrophysical plasma, there is a system of two vector equations:

$$\begin{cases} \frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) + \eta \nabla^2 \vec{\mathbf{B}} \\ \frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{F}} - \frac{1}{\rho} \vec{\nabla} \left(P + \frac{B^2}{2\mu_0} \right) + \frac{\vec{\mathbf{B}} \cdot \vec{\nabla}}{\mu_0 \rho} \vec{\mathbf{B}} \end{cases}$$

The usual goal is to solve for $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. MHD is ideal if $\eta \to 0$. This equation has SI units, for Gaussian, we use the transformation $\frac{|\vec{\mathbf{B}}_{[G]}|}{|\vec{\mathbf{B}}_{[SI]}|} = \sqrt{\frac{4\pi}{\mu_0}}$, which leaves the first equation invariant and the second one, with $\mu_0 \to 4\pi$.

Alfvén's theorem

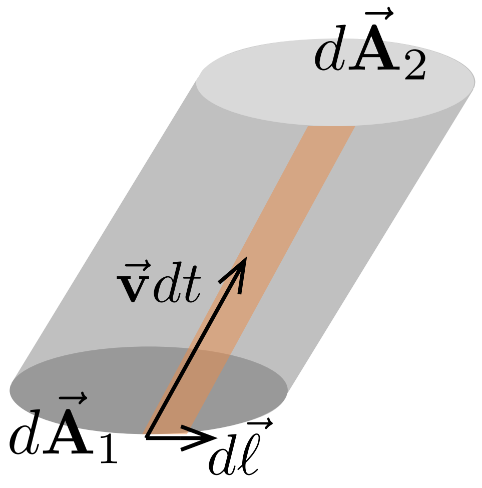
Magnetic Reynolds number: Comparison of the two terms of the induction equation: $\mathscr{R}_M \approx \dfrac{vB/L}{\eta B/L^2} \approx \dfrac{vL}{\eta}$. For an astrophysical plasma, L is very large, compared to a laboratory plasma. Then, the induction equation can be written as $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \eta \nabla^2 \vec{\mathbf{B}}$ for a laboratory plasma and $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ for an astrophysical plasma.

Alfvén's theorem of flux freezing: The Lagrangian derivative of the magnetic flux is $\dfrac{d\Phi}{dt} = \displaystyle\int_S \dfrac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} + \int \vec{\mathbf{B}} \cdot \dfrac{d}{dt} d\vec{\mathbf{A}}$. The change in the differential of area can be computed by noticing that its motion sweeps a cylinder (the full area to integrate is composed of small loops that delimit dA). Since the integral of the vector area of a closed surface is zero, $d\vec{\mathbf{A}}_2 - d\vec{\mathbf{A}}_1 - dt \displaystyle\oint \vec{\mathbf{v}} \times d\vec{\ell} = 0$. Then,

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} + \oint_C \vec{\mathbf{B}} \times \vec{\mathbf{v}} \cdot d\vec{\ell}$$

$$\implies \frac{d\Phi}{dt} = \int_S \left[\frac{\partial \vec{\mathbf{B}}}{\partial t} - \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \right] \cdot d\vec{\mathbf{A}} = 0.$$

Note that if the magnetic field is static (time derivative = 0), a configuration that works is a velocity field that is parallel to the magnetic field. The flux freezing theorem means that the magnetic flux is conserved and that if one moves the plasma, the magnetic field lines must also move with it (it's frozen).



Magnetic waves

Consider a static ($\vec{\mathbf{v}} = \vec{0}$) plasma threaded by a magnetic field like in the first figure on the right. Now imagine there is a small perturbation $\vec{\mathbf{v}}$ perpendicular to the original magnetic field, accompanied by a small perturbation of the magnetic field also in the same direction. Using perturbation theory like we did before, we find that the perturbation propagates as a wave (Alfvén wave) along the field lines with wave vector $\overrightarrow{\mathbf{k}}$ and speed $v_A := B_0/\sqrt{\mu_0\rho_0}$ (the subindex 0 means the unperturbed value). For this to happen, perturbation theory requires the perturbed velocity to be $|\vec{\mathbf{v}}| < v_a$ and we say that the the flow is sub-Alfvénic. If the perturbed velocity is larger than the Alfvén velocity ($|\vec{\mathbf{v}}| > v_a$), then we don't have an Alfvén wave anymore: the flow becomes super-Alfvénic and the velocity field simply drags the magnetic field lines with it without leaving them a chance to exert a restoring force and create a wave anymore. In a sub-Alfvénic flow, the magnetic field guides the flow (any deviation is restored by propagating an Alfvén wave), and in a super-Alfvénic flow, the velocity field drags the magnetic field lines.

Very simple derivation of the Alfvén speed: we take an order-of-magnitude view of the momentum equation ignoring external forces and the thermal pressure gradient. If we replace the time derivatives by $1/t$ (a characteristic propagation time of the perturbation) and the gradients by $1/L$ (a characteristic propagation distance of the perturbation) and think in 1D (imagining that we know already that the perturbation propagates along the magnetic field line), we get $\rho\dfrac{v}{t} \sim \dfrac{1}{L}\dfrac{B^2}{\mu_0}$. Substituting $v \sim L/t$ (the characteristic speed of propagation of the perturbation), we arrive at $v^2 \sim \dfrac{B^2}{\mu_0\rho} := v_A^2$.

Magnetoacoustic waves: if we don't ignore the thermal pressure gradient, the perturbation produces a combination of Alfvén and sound waves.