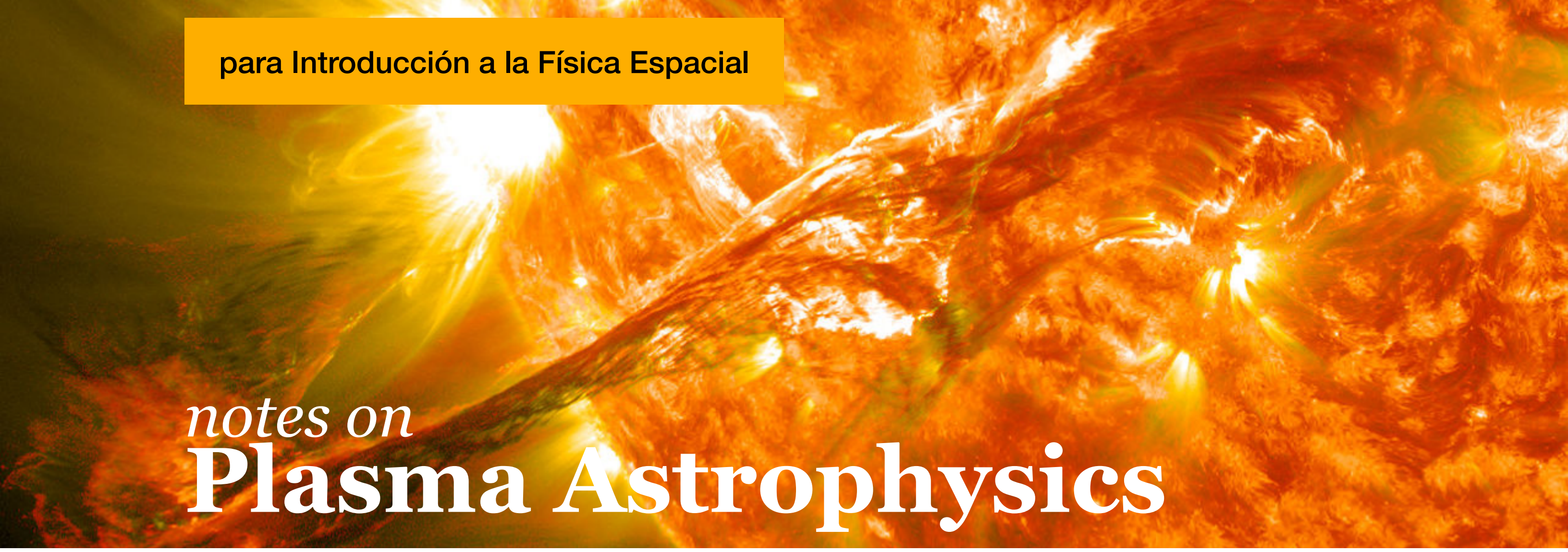
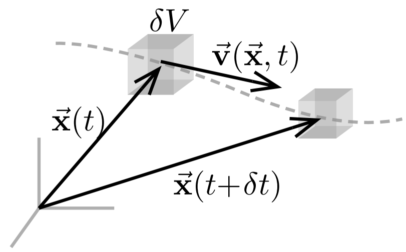


para Introducción a la Física Espacial

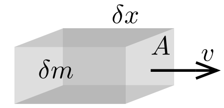
notes on Plasma Astrophysics

Fluid dynamics



The thermodynamic state of a small volume of fluid δV is given by its density, $\rho(\vec{\mathbf{x}}, t)$ and its temperature $T(\vec{\mathbf{x}}, t)$. Let $\vec{\mathbf{v}}(\vec{\mathbf{x}}, t)$ be the velocity of the small volume located in $\vec{\mathbf{x}}$ at time t.

Time derivatives: Let $\vec{\mathbf{x}}(t+\delta t) = \vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t$. For some quantity $Q(\vec{\mathbf{x}}, t)$, the time derivative is $\dfrac{dQ}{dt} = \lim_{\delta t \to 0} \dfrac{Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t, t + \delta t) - Q(\vec{\mathbf{x}}, t)}{\delta t}$. But making a Taylor expansion, $Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t, t + \delta t) = Q(\vec{\mathbf{x}}, t) + \delta t \dfrac{\partial Q}{\partial t} + \delta t\, \vec{\mathbf{v}} \cdot \vec{\nabla} Q$, which gives us $\dfrac{dQ}{dt} = \dfrac{\partial Q}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} Q$. The total derivative is called the *Lagrangian derivative* and the partial one, the *Eulerian derivative*.

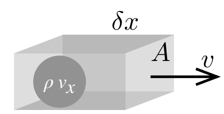


Continuity equation: for the change of mass, $\dfrac{\delta m}{\delta t} = \dfrac{\rho \delta V}{\delta t} = \dfrac{\rho A \delta x}{\delta t} = \rho A v$. In general, however, $\dfrac{\partial}{\partial t}\displaystyle\int \rho\, dV = -\oint \rho \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$. Using the Gauss's theorem, $\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) = 0$.

Euler equation (momentum equation): Newton's second law: $\rho \delta V \dfrac{d\vec{\mathbf{v}}}{dt} = \delta \vec{\mathbf{F}}_{\text{body}} + \delta \vec{\mathbf{F}}_{\text{surface}}$. But $\delta \vec{\mathbf{F}}_{\text{body}} = \rho \delta V \vec{\mathbf{F}}$ (where $\mathscr{D}[\vec{\mathbf{F}}] = F/M$, i.e., specific force = acceleration) and $\delta \vec{\mathbf{F}}_{\text{surface}} = -\displaystyle\oint P\, d\vec{\mathbf{A}} = -\int \vec{\nabla} P\, dV$. Inserting everything and changing the Lagrangian derivative, $\dfrac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = -\dfrac{1}{\rho} \vec{\nabla} P + \vec{\mathbf{F}}$.

Advection vs convection: convection is the movement of a fluid mainly due to density gradients created by thermal gradients; advection is the more general transport of material or physical quantity by the velocity of the fluid.

Advection equation: the general equation $\dfrac{\partial \psi}{\partial t} + \vec{\nabla} \cdot (\psi \vec{\mathbf{v}}) = 0$ states the advection for a conserved quantity described by a scalar field ψ due to the transport in the velocity field $\vec{\mathbf{v}}$. If $\vec{\nabla} \cdot \vec{\mathbf{v}} = 0$ (incompressible flow / solenoidal) $\Longrightarrow \dfrac{\partial \psi}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} \psi = 0$.



Conservative form of the momentum equation: consider a element of fluid that contains a momentum in the x direction $\delta \mathscr{P}_x = \rho v_x \delta V$. Then, let's consider how this momentum changes in time in a similar way to what we did for the continuity equation: $\dfrac{\delta \mathscr{P}_x}{\delta t} = \dfrac{\rho v_x \delta V}{\delta t} = \rho v_x A \dfrac{\delta x}{\delta t} = \rho v_x A v$. Here, v_x is the *x*-component of the velocity *inside of* the fluid element (used to compute the momentum of said fluid element), and v is the velocity across the wall, that is, the velocity that transports (advects) the momentum out of the fluid element across the wall A. They are the same velocity field, just measured at different positions (they reduce to the same vector value when $\delta V \to 0$). Then, in general for all the walls, $\frac{\partial}{\partial t}\int \rho v_x dV = -\oint \rho v_x \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$ (this means in words that any momentum decrease inside of the fluid element in the *x*-direction must happen because it is lost through the walls of the element). Using Gauss's theorem, $\implies \frac{\partial}{\partial t}(\rho v_x) + \vec{\nabla} \cdot (\rho v_x \vec{\mathbf{v}}) = 0$ (momentum is advected and conserved). This is true if there are no forces. If there are forces, they act as the sources of the equation and we have $\frac{\partial}{\partial t}(\rho v_x) + \vec{\nabla} \cdot (\rho v_x \vec{\mathbf{v}}) = -\frac{\partial P}{\partial x} + \rho F_x$. One can show that this equation is equivalent to the Euler equation. There are similar equations for the *y* and *z* directions.

Conservation of energy: the same reasoning can be applied for the scalar field defined by the sum of the (volumetric) kinetic energy $\frac{1}{2}\rho v^2$ and the (volumetric) internal energy $\rho\epsilon$ and we obtain an equation of conservation of energy, where the sources can be: heating/cooling of the gas by thermal contraction/expansion, mechanical heating/cooling by external forces, heating/cooling by other processes such as radiation (absorption/emission), etc. In addition, we need an *equation of state* (relation between pressure and density, for example, the ideal gas law).

Hydrodynamical perturbations

Basic equations: The equations of hydrodynamics are: (a) the continuity equation, (b) the Euler equation. In an astrophysical plasma, $\vec{\mathbf{F}}$ (force per unit mass) is the gravitational field, $\vec{\mathbf{F}} = -\vec{\nabla}\Phi$, that satisfies the Poisson equation $\vec{\nabla}^2\Phi = 4\pi G\rho$ (c).

$$\begin{cases} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) = 0 & \text{(a)} \\ \frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = -\frac{1}{\rho}\vec{\nabla}P + \vec{\mathbf{F}} & \text{(b)} \\ \vec{\nabla}^2\Phi = 4\pi G\rho & \text{(c)} \\ \frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0 & \text{(d)} \end{cases}$$

Thermodynamics: small perturbations of pressure and density imply changes in temperature $\Longrightarrow$ it's not an isothermal process. However, small perturbations don't cause significant heat exchange with surroundings $\therefore$ adiabatic process with equation $PV^\gamma = \text{const} \implies d/dt(P/\rho^\gamma) = 0$ (d). Also, the equation of state of an ideal gas holds ($PV = Nk_BT$).

Perturbations: the plasma is in an initial equilibrium state $\{\vec{\mathbf{v}}_0 = 0, \rho_0, P_0, \Phi_0\}$ (at the beginning there is no displacement and then, no velocity). The perturbed values are then $\{\vec{\mathbf{v}} = \vec{\mathbf{v}}_1, \rho = \rho_0 + \rho_1, P = P_0 + P_1, \Phi = \Phi_0 + \Phi_1\}$ where $\rho_1 \ll \rho_0$, $P_1 \ll P_0$, $|\Phi_1| \ll |\Phi_0|$.

0th-order results:

(a) $\Longrightarrow \partial_t \rho_0 = 0 \implies \rho = \text{const.}$ (a.0) (b) $\implies \vec{\nabla}P_0 = -\rho_0\vec{\nabla}\Phi_0$. (b.0)
(c) $\nabla^2\Phi_0 = 4\pi G\rho_0$. (c.0) (d) $\implies P_0/\rho_0^\gamma = \text{const.}$ (d.0)

Note that a self-gravitating, uniform, infinite gas cannot exist, since if P_0 is constant everywhere, (b.0) $\implies \Phi_0$: const, but then, (c.0) $\implies \rho_0 = 0$.

1st-order results:

(d) $\implies \frac{d}{dt}\left[\frac{P_0 + P_1}{(\rho_0 + \rho_1)^\gamma}\right] = 0$, and with (d.0), $\frac{P_0 + P_1}{(\rho_0 + \rho_1)^\gamma} = \frac{P_0}{\rho_0^\gamma} \implies 1 + \frac{P_1}{P_0} = \left(1 + \frac{\rho_1}{\rho_0}\right)^\gamma$. Since $\rho_1/\rho_0 \ll 1$, we can expand the rhs, so that $1 + \frac{P_1}{P_0} \approx 1 + \gamma\frac{\rho_1}{\rho_0} \Longrightarrow P_1 = \left(\frac{\gamma P_0}{\rho_0}\right)\rho_1 := c_s^2\,\rho_1$ **(d.1)**.

(a), (a.0) $\implies \partial_t \rho_1 + \vec{\nabla}[(\rho_0 + \rho_1)\vec{\mathbf{v}}_1] = 0$. The term $\rho_1 \vec{\mathbf{v}}_1$ is smaller than first order, so it goes to zero, and ρ_0 is constant. This implies $\partial_t \rho_1 + \rho_0 \vec{\nabla} \cdot \vec{\mathbf{v}}_1 = 0$ **(a.1)**.

(b) $\implies (\rho_0 + \rho_1)\left[\partial_t \vec{\mathbf{v}}_1 + (\vec{\mathbf{v}}_1 \cdot \vec{\nabla})\vec{\mathbf{v}}_1\right] = -\vec{\nabla}(\underline{P_0} + P_1) - \vec{\nabla}(\underline{\Phi_0} + \Phi_1)(\underline{\rho_0} + \rho_1)$, but the underlined terms cancel each other due to (b.0). The second term of the Lagrangian derivative is small, and so are the other terms involving ρ_1. Then, we have $\rho_0 \partial_t \vec{\mathbf{v}}_1 = -\vec{\nabla} P_1 - \rho_0 \vec{\nabla} \Phi_1$. Using (d.1), this becomes $\rho_0 \partial_t \vec{\mathbf{v}}_1 = -c_s^2 \vec{\nabla} \rho_1 - \rho_0 \vec{\nabla} \Phi_1$ **(b.1)**.

(c), (c.0) $\implies \vec{\nabla}^2 \Phi_1 = 4\pi G \rho_1$ **(c.1)**.

Sound equation: if gravity is negligible, for example, in the atmosphere, we have, for eq. (b.1), and taking the divergence in both sides, $\vec{\nabla} \cdot \left[\rho_0 \partial_t \vec{\mathbf{v}}_1 = -c_s^2 \vec{\nabla} \rho_1\right]$. Rearranging the lhs, we can have $\partial_t(\rho_0 \vec{\nabla} \cdot \vec{\mathbf{v}}_1)$, that we can substitute using (a.1), so that we have $\dfrac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \vec{\nabla}^2 \rho_1$. This is a wave equation, and so we know that c_s has to be the speed of sound. This means that perturbations travel with the speed of sound (that was evaluated in the 0th order).

Periodic perturbations: we take the equations (a.1), (b.1) and (c.1). If the small perturbations are periodic, we can write solutions for each variable h of the form $h_{\max} e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} - \omega t)}$, which is the same as taking the Fourier transform of each equation; moreover, $\mathscr{F}\{\partial_t\} = -i\omega$, $\mathscr{F}\{\vec{\nabla}\} = i\vec{\mathbf{k}}$. Then, the system of equations becomes: $\omega \rho_1 = \rho_0 \vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_1$ **(a1F)** $\omega \rho_0 \vec{\mathbf{v}}_1 = \vec{\mathbf{k}} c_s^2 \rho_1 + \vec{\mathbf{k}} \rho_0 \Phi_1$ **(b1F)** $\vec{\mathbf{k}}^2 \Phi_1 = -4\pi G \rho_1$ **(c1F)**

Jeans instability: (a1F), ($\mathbf{v}_1$ from b1F) $\implies \omega^2 \rho_1 = \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} c_s^2 \rho_1 + \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} \rho_0 \Phi_1$. (c1F), ($\not{\rho_1}$) $\implies \omega^2 = k^2 c_s^2 - 4\pi G \rho_0$. We write this dispersion relation as $\omega^2 = c_s^2 \left[k^2 - \dfrac{4\pi G \rho_0}{c_s^2}\right] := c_s^2(k^2 - k_J^2)$. Notice that ω is real only if $k_J < k$ (oscillation), but if $k_J > k$, ω becomes complex and the exponential on the temporal part of each variable becomes real, leading to an exponential grow. This means that if the size of the perturbation is larger than $\lambda_J = 2\pi / k_J$ and self-gravity overpowers acoustic waves. The mass that corresponds to λ_J is called the *Jeans mass*, and is calculated as $M = \dfrac{4}{3}\pi \lambda_J^3 \rho_0$. Using (d.1) and the equation of state of the ideal gas, as well as the approximation $\gamma \approx 1$ (slow perturbations are approx. isothermal), then, $M_J = \dfrac{4}{3}\pi^{5/2} \rho_0^{-1/2} \left(\dfrac{k_B T}{G m}\right)^{3/2}$ (m is the mass of a molecule).

Waves vs instabilities: notice that in the case of the sound equation, we wanted a perturbation small enough to be propagated as a wave. On the contrary, in the case of the Jeans instability, we wanted a perturbation large enough to grow indefinitely and form an instability.

Equations of magneto-hydrodynamics

Curl of a cross product: $\vec{\nabla} \times (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = (\vec{\mathbf{B}} \cdot \vec{\nabla})\vec{\mathbf{A}} - \vec{\mathbf{B}}(\vec{\nabla} \cdot \vec{\mathbf{A}}) + (\vec{\nabla} \cdot \vec{\mathbf{B}})\vec{\mathbf{A}} - (\vec{\mathbf{A}} \cdot \vec{\nabla})\vec{\mathbf{B}}$. First, we apply the product rule (marking which vector $\vec{\nabla}$ is acting on) and then we expand using the BAC-CAB rule.

Gradient of a dot product: $\vec{\nabla}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) = \dot{\vec{\nabla}}(\dot{\vec{\mathbf{A}}} \cdot \vec{\mathbf{B}}) + \dot{\vec{\nabla}}(\vec{\mathbf{A}} \cdot \dot{\vec{\mathbf{B}}})$ (the dot marks where the derivative acts on). But applying the BAC-CAB rule to the cross product of a curl, $\vec{\mathbf{A}} \times (\vec{\nabla} \times \vec{\mathbf{B}}) = \dot{\vec{\nabla}}(\vec{\mathbf{A}} \cdot \dot{\vec{\mathbf{B}}}) - (\vec{\mathbf{A}} \cdot \vec{\nabla})\vec{\mathbf{B}}$, one of the terms appear. Making $\vec{\mathbf{A}} \leftrightarrows \vec{\mathbf{B}}$ and substituting, we find $\vec{\nabla}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) = (\vec{\mathbf{A}} \cdot \vec{\nabla})\vec{\mathbf{B}} + (\vec{\mathbf{B}} \cdot \vec{\nabla})\vec{\mathbf{A}} + \vec{\mathbf{A}} \times (\vec{\nabla} \times \vec{\mathbf{B}}) + \vec{\mathbf{B}} \times (\vec{\nabla} \times \vec{\mathbf{A}})$.

Ohm's law: $\vec{\mathbf{j}} = \sigma \vec{\mathbf{E}}$ or $\eta \vec{\mathbf{j}} = \vec{\mathbf{E}}$, but since the magnetic field also contributes to the force on charges, $\vec{\mathbf{j}} = \sigma(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$.

Ampère-Maxwell equation: for astrophysical plasmas with $v \ll c$, we ignore the displacement current term. So, $\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{j}}$.

Electric field: combining Ampère's and Ohm's laws, $\vec{\mathbf{E}} = \dfrac{\vec{\nabla} \times \vec{\mathbf{B}}}{\mu_0 \sigma} - \vec{\mathbf{v}} \times \vec{\mathbf{B}}$. For astrophysical plasmas, $\vec{\mathbf{B}}$ is more relevant than $\vec{\mathbf{E}}$, because charges are well mixed.

Induction equation: Faraday's law: $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = -\vec{\nabla} \times \vec{\mathbf{E}}$. Substituting the electric field, $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) + \eta \nabla^2 \vec{\mathbf{B}}$, where $\eta = 1/(\mu_0 \sigma)$. [Application of the "BAC-CAB" rule for gradients, one of the terms will be $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$. $\vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ is called the *advection* term.]

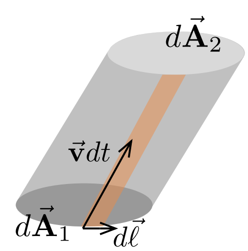
Euler equation: modification: $\vec{\mathbf{F}} \to \vec{\mathbf{F}} + \dfrac{1}{\rho} \vec{\mathbf{j}} \times \vec{\mathbf{B}}$. Substituting $\vec{\mathbf{j}}$ (with Ampère's law), using the gradient of a dot product when both vectors are equal, $\dfrac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{F}} - \dfrac{1}{\rho} \vec{\nabla}\left(P + \dfrac{B^2}{2\mu_0}\right) + \dfrac{\vec{\mathbf{B}} \cdot \vec{\nabla}}{\mu_0 \rho} \vec{\mathbf{B}}$. The first new term is the magnetic field pressure and the second term, the magnetic tension force, which tends to straighten magnetic field lines.

Summary: for an astrophysical plasma, there is a system of two vector equations:

$$\begin{cases} \frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) + \eta \nabla^2 \vec{\mathbf{B}} \\ \frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{F}} - \frac{1}{\rho} \vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{\vec{\mathbf{B}} \cdot \vec{\nabla}}{\mu_0 \rho} \vec{\mathbf{B}} \end{cases}$$

The usual goal is to solve for $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. MHD is *ideal* if $\eta \to 0$. This equation has SI units, for Gaussian, we use the transformation $\dfrac{|\vec{\mathbf{B}}_{[G]}|}{|\vec{\mathbf{B}}_{[SI]}|} = \sqrt{\dfrac{4\pi}{\mu_0}}$, which leaves the first equation invariant and the second one, with $\mu_0 \to 4\pi$.

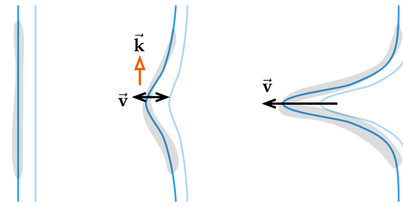
Flux freezing



Magnetic Reynolds number: Comparison of the two terms of the induction equation: $\mathscr{R}_M \approx \dfrac{vB/L}{\eta B/L^2} \approx \dfrac{vL}{\eta}$. For an astrophysical plasma, L is very large, compared to a laboratory plasma. Then, the induction equation can be written as $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \eta \nabla^2 \vec{\mathbf{B}}$ for a laboratory plasma and $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ for an astrophysical plasma.

Alfvén's theorem of flux freezing: The Lagrangian derivative of the magnetic flux is $\dfrac{d\Phi}{dt} = \displaystyle\int_S \dfrac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} + \int \vec{\mathbf{B}} \cdot \dfrac{d}{dt} d\vec{\mathbf{A}}$. The change in the differential of area can be computed by noticing that its motion sweeps a cylinder (the full area to integrate is composed of small loops that delimit dA). Since the integral of the vector area of a closed surface is zero, $d\vec{\mathbf{A}}_2 - d\vec{\mathbf{A}}_1 - dt \oint \vec{\mathbf{v}} \times d\vec{\ell} = 0$. Then, $\dfrac{d\Phi}{dt} = \displaystyle\int_S \dfrac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} + \oint_C \vec{\mathbf{B}} \times \vec{\mathbf{v}} \cdot d\vec{\ell} \implies \dfrac{d\Phi}{dt} = \int_S \left[\dfrac{\partial \vec{\mathbf{B}}}{\partial t} - \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})\right] \cdot d\vec{\mathbf{A}} = 0$. Note that if the magnetic field is static (time derivative = 0), a configuration that works is a velocity field that is parallel to the magnetic field. The flux freezing theorem means that the magnetic flux is conserved and that if one moves the plasma, the magnetic field lines must also move with it (it's frozen).

Alfvén waves



Consider a static ($\vec{\mathbf{v}} = \vec{0}$) plasma threaded by a magnetic field like in the first figure on the right. Now imagine there is a small perturbation $\vec{\mathbf{v}}$ perpendicular to the original magnetic field, accompanied by a small perturbation of the magnetic field also in the same direction. Using perturbation theory like we did before, we find that the perturbation propagates as a wave (**Alfvén wave**) along the field lines with wave vector $\overrightarrow{\mathbf{k}}$ and speed $v_A := B_0/\sqrt{\mu_0\rho_0}$ (the subindex 0 means the unperturbed value). For this to happen, perturbation theory requires the perturbed velocity to be $|\vec{\mathbf{v}}| < v_a$ and we say that the the flow is **sub-Alfvénic**. If the perturbed velocity is larger than the Alfvén velocity ($|\vec{\mathbf{v}}| > v_a$), then we don't have an Alfvén wave anymore: the flow becomes **super-Alfvénic** and the velocity field simply drags the magnetic field lines with it without leaving them a chance to exert a restoring force and create a wave anymore. In a sub-Alfvénic flow, the magnetic field guides the flow (any deviation is restored by propagating an Alfvén wave), and in a super-Alfvénic flow, the velocity field drags the magnetic field lines.

Very simple derivation of the Alfvén speed: we take an order-of-magnitude view of the momentum equation ignoring external forces and the thermal pressure gradient. If we replace the time derivatives by $1/t$ (a characteristic propagation time of the perturbation) and the gradients by $1/L$ (a characteristic propagation distance of the perturbation) and think in 1D (imagining that we know already that the perturbation propagates along the magnetic field line), we get $\rho\frac{v}{t} \sim \frac{1}{L}\frac{B^2}{\mu_0}$. Substituting $v \sim L/t$ (the characteristic speed of propagation of the perturbation), we arrive at $v^2 \sim \frac{B^2}{\mu_0\rho} := v_A^2$.

Magnetoacoustic waves: if we don't ignore the thermal pressure gradient, the perturbation produces a combination of Alfvén and sound waves.

Non-ideal MHD effects

To summarize what our discussion on magnetic flux: we have Faraday's law:

$$\frac{\partial\overrightarrow{\mathbf{B}}}{\partial t} = -\vec{\nabla}\times\overrightarrow{\mathbf{E}}$$

and then, we have several contributions (that must be added) to the electric field $\overrightarrow{\mathbf{E}}$.

a) $\overrightarrow{\mathbf{E}} = -\vec{\mathbf{v}}\times\overrightarrow{\mathbf{B}} \Longrightarrow$ conservation of magnetic flux (advective term)

b) $\overrightarrow{\mathbf{E}}_O = \eta\vec{\mathbf{j}}$: Ohmic dissipation (we substitute $\vec{\mathbf{j}} = \vec{\nabla}\times\overrightarrow{\mathbf{B}}/\mu_0$ to get the equations we discussed above)

Now we add new contributions:

Ambipolar drift: consider a partly ionized plasma made up of a fully ionized part (density ρ_p, flow velocity $\vec{\mathbf{v}}_p$, pressure P_p) and a neutral part (density ρ_n, flow velocity $\vec{\mathbf{v}}_n$, pressure P_n). The velocity of the center of mass is $\vec{\mathbf{v}} := (\rho_p\vec{\mathbf{v}}_p + \rho_n\vec{\mathbf{v}}_n)/\rho$ (ρ: total density). Since the ions (negative and positive) have different masses, this creates a net current $\vec{\mathbf{j}}$ (normal plasma treatment); however, the neutral species also has a different mass, and it's not affected by the magnetic and electric fields, which means that there is a drift velocity $\vec{\mathbf{v}}_D = \vec{\mathbf{v}}_p - \vec{\mathbf{v}}_n$ between the plasma and the neutral species. Ambipolar drift redistributes magnetic flux, which can trigger star formation; it affects short wavelength interstellar interstellar turbulence, the structure of interstellar shocks, and the nature of magnetic reconnection.

Strong coupling approximation: the plasma and neutrals are coupled dynamically and thermally, and if the medium is weakly ionized, one can treat it as a single fluid. We ignore the plasma inertia ($\implies \rho \approx \rho_n$, $\vec{\mathbf{v}} \approx \vec{\mathbf{v}}_n$), and so, the momentum equation is simply the same as the plasma (b). For the induction equation (d), however, only the plasma is affected, so, $\partial_t \vec{\mathbf{B}} = \vec{\nabla} \times (\vec{\mathbf{v}}_p \times \vec{\mathbf{B}})$ $\implies \partial_t \vec{\mathbf{B}} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) + \vec{\nabla} \times (\vec{\mathbf{v}}_D \times \vec{\mathbf{B}})$. The plasma and neutrals follow almost the same dynamics, but the field is not perfectly frozen into the medium because of ambipolar drift.

Multiple fluids: it is necessary to take a multi-fluid approach in high freq. waves, small scale turbulence, turbulent dynamos, shocks, magnetic reconnection.

Heavy ion approximation: suitable for problems in which the plasma-neutral coupling is quite strong, such as studies of large-scale turbulence in dense molecular clouds. It's less successful for weakly coupled problems (plasma and gas follow their own dynamics). For example, in molecular clouds, the ratio of plasma pressure to magnetic pressure is much smaller than assumed in the heavy ion approximation.

With those approximations, ambipolar diffusion, we can define another contribution to the electric field:

c) $\vec{\mathbf{E}}_{AD} \propto \eta_{AD} \vec{\mathbf{B}} \times (\vec{\mathbf{j}} \times \vec{\mathbf{B}})$

where η_{AD} is the *ambipolar diffusivity* (equivalent to the Ohmic resistivity and dependent on the microphysics of the plasma, for example the collision rates between charged particles and neutrals)

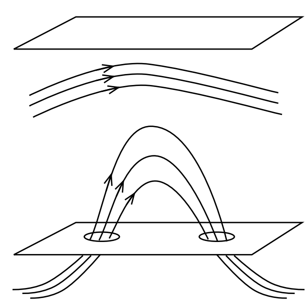
Hall effect: the current of the plasma, while immersed in the magnetic field of the plasma itself induces an electric field

d) $\vec{\mathbf{E}}_H \propto \eta_H \vec{\mathbf{j}} \times \vec{\mathbf{B}}$

Therefore, the induction equation $\frac{\partial \vec{\mathbf{B}}}{\partial t} = -\vec{\nabla} \times \vec{\mathbf{E}}$ becomes a kind of conservation equation (but using a curl instead of a divergence) for the magnetic flux with an advective term $\vec{\mathbf{E}} = -\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ (the velocity field advects the magnetic flux) and several sinks (ways of getting rid of the magnetic flux) via Ohmic dissipation, ambipolar diffusion and the Hall effect.

Magnetic buoyancy

Consider a horizontal cylindrical region of concentrated magnetic field, called *magnetic flux tube*. The flux is frozen, which means that both the plasma and the magnetic field respond immediately to changes. If there is hydrostatic balance, $P_{\text{ext}} = P_{\text{int}} + \frac{B^2}{2\mu_0}$. Then, $P_{\text{int}} < P_{\text{ext}}$. But if the gas is ideal, $P \propto \rho T$, and if the temperatures are the same, $\rho_{\text{int}} < \rho_{\text{ext}}$, making the whole section of the tube buoyant.

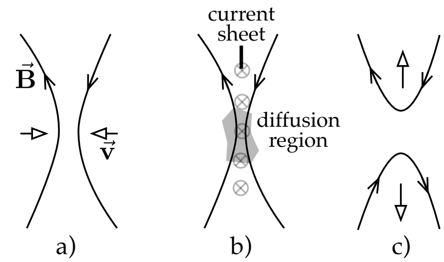


Magnetic reconnection

c) Opposing magnetic field lines approach each other.

d) When the lines are too close, $\eta \nabla^2 \overrightarrow{\mathbf{B}}$ can no longer be ignored and a diffusion region is created. The magnetic field configuration requires the existence of a current sheet.

e) Magnetic field lines are reconfigured. The process makes a conversion of magnetic energy into kinetic energy. Plasma is propelled. The process is sustainable: lowering the magnetic field in the center also lowers the pressure, causing plasma to get "sucked in" and the process to continue.



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