Problem: compute the Fourier transform of the following functions using the definition:

1) The Dirac delta $\delta(t-t_0)$

$$\mathcal{F}\{\delta(t-t_0)\}(\omega) = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-i\omega t}dt = e^{-i\omega t_0} = F(\omega)$$

2) The step function with a decreasing exponential

$$f:\mathbb{R}\to\mathbb{R}, \quad f(t) = \begin{cases} e^{-at}, & t\geq 0 \\ 0, & t<0 \end{cases} = e^{-at}H(t) \quad \text{with } a \geq 0$$

$$\mathcal{F}\{e^{-at}H(t)\} = \int_0^{\infty} e^{-(a+i\omega)t}dt = \lim_{b\to\infty}\int_0^b e^{-(a+i\omega)t}dt$$

let $u = -(a+i\omega)t \Rightarrow du = -(a+i\omega)dt$

$$\Rightarrow \lim_{b\to\infty} \frac{-1}{a+i\omega}\left[e^{-(a+i\omega)b} - e^0\right] = \frac{1}{a+i\omega} = F(\omega)$$

$$\underbrace{\underbrace{e^{-a\infty}}_{\text{real decreasing envelope}} \cdot e^{-i\omega\infty}}_{\to 0}$$

limit doesn't exist (oscillatory behavior)

3) The Heaviside or step function

$$H(t):\mathbb{R}\to\mathbb{R}, \quad H(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

$$\mathcal{F}\{H(t)\}(\omega) = \int_0^{\infty} e^{-i\omega t}dt = \lim_{b\to\infty}\frac{-1}{i\omega}\left[e^{-i\omega t}\right]_0^b$$

the limit $e^{-i\omega\infty}$ doesn't exist (oscillations at infinity). But we can use the result from the previous example, thinking of the Heaviside function as a "limiting function" of the decreasing exponential step function when $a \to 0$

Then,

$$\mathcal{F}\{H(t)\}(\omega) = \frac{1}{i\omega}, \quad \text{iff } \omega \neq 0.$$

But if we compute the inverse transform by the definition,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\omega}e^{i\omega t}d\omega$$

we see that we must include the case $\omega = 0$.

$$\frac{1}{2\pi}\lim_{b\to 0^+}\int_b^{\infty}\frac{1}{i\omega}e^{i\omega t}d\omega + \frac{1}{2\pi}\lim_{b\to 0^-}\int_{-\infty}^{b}\frac{1}{i\omega}e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{0^-}^{0^+} f_1(\omega)e^{i\omega t}d\omega$$

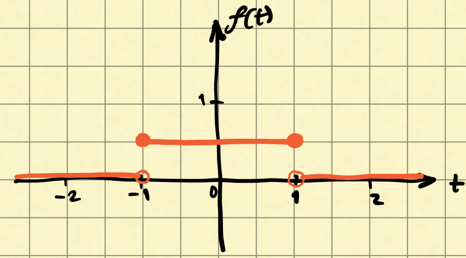
so we need a function f_1 that only exists at $\omega=0$, such that the last integral is finite. This is the Dirac delta. Then we have

$$\mathcal{F}\{H(t)\}(\omega) = \text{P.V.}\frac{1}{i\omega} + \pi\delta(\omega)$$

where p.v. means Cauchy's principal value.

4) The rectangular function

$$f:\mathbb{R}\to\mathbb{R}, \quad f(t)=\begin{cases} 1/2, & |t|\le 1 \\ 0, & |t|>1 \end{cases}$$



$$\mathcal{F}\{f(t)\}(\omega) = \frac{1}{2}\int_{-1}^{1} e^{-i\omega t}\,dt$$

- $\omega=0$:

$$= \frac{1}{2}\int_{-1}^{1} dt = 1$$

- $\omega\neq 0$:

$$= \frac{1}{2}\int_{-1}^{1} e^{-i\omega t}\,dt = \frac{1}{2}\left[\frac{-1}{i\omega}e^{-i\omega t}\right]_{-1}^{1} = \frac{1}{\omega}\left[\frac{e^{i\omega}-e^{-i\omega}}{i}\right] = \frac{\sin\omega}{\omega}$$

$$\therefore\ \mathcal{F}\{f(t)\}(\omega) = \begin{cases} 1, & \omega=0 \\ \dfrac{\sin\omega}{\omega}, & \omega\neq 0 \end{cases} := \operatorname{sinc}\omega .$$

Problem: show the property of similarity of the Fourier transform.

$$\mathcal{F}\{f(ct)\} = \frac{1}{|c|}F\left(\frac{\omega}{c}\right)$$

$$\mathcal{F}\{f(ct)\}(\omega) = \int_{-\infty}^{\infty} f(ct)\,e^{-i\omega t}\,dt = \left(\begin{array}{l}\text{let } u=ct \\ \Rightarrow du = c\,dt\end{array}\right) = \int_{-\infty}^{\infty} f(u)\,e^{-i\omega u/c}\,\frac{du}{|c|} = \frac{1}{|c|}F\left(\frac{\omega}{c}\right)$$

limits don't change order only if there is an absolute value.

Problem: show the property of displacement in time of the Fourier transform.

$$\mathcal{F}\{f(t-a)\} = e^{-i\omega a}F(\omega)$$

$$\int_{-\infty}^{\infty} f(t-a)\,e^{-i\omega t}\,dt = \left(\begin{array}{l}\text{let } u=t-a \\ t=u+a \\ du=dt\end{array}\right) = \int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u+a)}\,du = \int_{-\infty}^{\infty} f(u)\,e^{-i\omega a}\cdot e^{-i\omega u}\,du$$

$$= e^{-i\omega a}F(\omega)$$

Problem: show the property of the derivative in frequencies of the Fourier transform.

$$\mathcal{F}\{t f(t)\} = iF'(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt \Rightarrow \frac{d}{d\omega}F(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt \Rightarrow F'(\omega) = \int_{-\infty}^{\infty} -i\,t f(t)\,e^{-i\omega t}\,dt$$

$$\Rightarrow iF'(\omega) = \int_{-\infty}^{\infty} t f(t)\,e^{-i\omega t}\,dt \Rightarrow \mathcal{F}\{t f(t)\} = iF'(\omega) .$$

Problem: Compute the Fourier transform of the function $e^{-at}\sin\omega_0 t H(t)$, $H(t)$: Heaviside

$\mathcal{F}\{e^{-at}\sin\omega_0 t\, H(t)\}$

$$\mathcal{F}\left\{e^{-at}\cdot\frac{1}{2i}\cdot(e^{i\omega_0 t}-e^{-i\omega_0 t})H(t)\right\}$$

$$=\frac{1}{2i}\mathcal{F}\{e^{-at}e^{i\omega_0 t}H(t)\}-\frac{1}{2i}\mathcal{F}\{e^{-at}e^{-i\omega_0 t}H(t)\}$$

De la tabla, sabemos que $\mathcal{F}\{e^{-at}H(t)\}=\frac{1}{a+i\omega}$. También, que

$\mathcal{F}\{e^{i\omega_0 t}f(t)\}=\mathcal{F}\{f(t)\}(\omega-\omega_0)=F(\omega-\omega_0)$

$$\frac{1}{i}=\frac{1}{\sqrt{-1}}=\frac{\sqrt{-1}}{\sqrt{-1}}\cdot\frac{1}{\sqrt{-1}}$$
$$=-\sqrt{-1}=-i$$

$$\Rightarrow\quad \frac{1}{2i}\,\frac{1}{a+i(\omega-\omega_0)}-\frac{1}{2i}\,\frac{1}{a+i(\omega+\omega_0)}$$

$$=\frac{-i}{2}\left[\frac{\cancel{a}+i(\omega+\omega_0)-\cancel{a}-i(\omega-\omega_0)}{[a+i(\omega-\omega_0)][a+i(\omega+\omega_0)]}\right]$$

$$=\frac{-i}{2}\,\frac{i\cancel{\omega}+i\omega_0-i\cancel{\omega}+i\omega_0}{a^2+ai(\omega+\cancel{\omega_0})+ai(\omega-\cancel{\omega_0})-(\omega^2-\omega_0^2)}$$

$$=\frac{-i}{\cancel{2}}\,\frac{\cancel{2}i\omega_0}{a^2+2ai\omega-\omega^2+\omega_0^2}$$

$$=\frac{\omega_0}{(a+i\omega)^2+\omega_0^2}$$

Problem: solve the differential equation

$$\ddot{x}+\omega_0^2x=0\quad,\quad\omega_0>0$$

which describes a harmonic oscillator using Fourier transforms.

$$\mathcal{F}\{\ddot{x}\}+\mathcal{F}\{\omega_0^2x\}=0$$
$$i^2\omega^2X(\omega)+\omega_0^2X(\omega)=0$$
$$\Rightarrow\quad(\omega_0^2-\omega^2)X(\omega)=0$$
$$\Rightarrow\quad(\omega^2-\omega_0^2)X(\omega)=0$$
$$\Rightarrow\quad(\omega+\omega_0)(\omega-\omega_0)X(\omega)=0$$

Por las propiedades de la delta,

si $X(\omega)=\delta(\omega-\omega_0)\cdot 2\pi$

$$\Rightarrow\quad 2\pi(\omega-\omega_0)\delta(\omega-\omega_0)=0$$

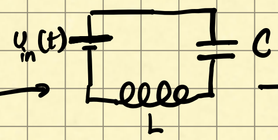
entonces,

$$X(\omega)=2\pi\delta(\omega-\omega_0)\cdot A$$
$$\Rightarrow\quad x(t)=\mathcal{F}^{-1}\{A2\pi\delta(\omega-\omega_0)\}$$
$$x(t)=Ae^{-i\omega_0 t}$$

Nota: una suma de deltas también es solución.

Problem: solve the differential equation

$$\ddot{x} + \omega_0^2 x = \cos \omega_1 t$$



$$\overset{\text{voltage}}{\ddot{v}(t)} + \underbrace{\frac{1}{LC}}_{\omega_0^2} v(t) = \overbrace{\cos(\omega_1 t)}^{U_{in}(t)/LC}$$

which represents a forced harmonic oscillator or a LC circuit, using Fourier transforms.

$$\mathcal{F}\{\ddot{x}\} + \mathcal{F}\{\omega_0^2 x\} = \mathcal{F}\{\cos \omega_1 t\}$$

$$i^2\omega^2 X(\omega) + \omega_0^2 X(\omega) = \pi\left[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)\right]$$

$$-\omega^2 X(\omega) + \omega_0^2 X(\omega) = \pi\left[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)\right]$$

$$X(\omega)(\omega_0^2 - \omega^2) = \pi\left[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)\right]$$

$$X(\omega) = \frac{\pi\left[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)\right]}{\omega_0^2 - \omega^2}$$

$$F(\omega) = \frac{1}{\omega_0^2 - \omega^2} \qquad G(\omega) = \left[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)\right]\cdot\pi$$

Convolución

$$x(t) = \mathcal{F}^{-1}\left[F(\omega)G(\omega)\right] = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,dt$$

en este caso, intentamos directamente

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega)G(\omega)\,d\omega$$

$$= \frac{\cancel{\pi}}{2\cancel{\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\cdot\frac{1}{\omega_0^2-\omega^2}\cdot\left[\delta(\omega-\omega_1)+\delta(\omega+\omega_1)\right]d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega t}\frac{1}{\omega_0^2-\omega^2}\,\delta(\omega-\omega_1)\,d\omega + \frac{1}{2}\int_{-\infty}^{\infty} e^{i\omega t}\frac{1}{\omega_0^2-\omega^2}\,\delta(\omega+\omega_1)\,d\omega$$

$$= \frac{1}{2}\left[e^{i\omega_1 t}\frac{1}{\omega_0^2-\omega_1^2} + e^{-i\omega_1 t}\frac{1}{\omega_0^2-\omega_1^2}\right] = \frac{\cos(\omega_1 t)}{\omega_0^2-\omega_1^2}$$

Additional properties of the Dirac delta

1. Relationship between the Dirac delta and the Kronecker delta. We can see the similarity of the definition when we consider what it does to a sum and an integral:

$$\sum_{i=-\infty}^{\infty} a_i \delta_{ij} = a_j \qquad \text{all the indexes vanish except for } i=j$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0) \qquad \text{all } x \text{ vanish except for } x = x_0$$

So, we say the Dirac Delta is the continuous analog of the Kronecker delta.

2. Different limits of integration. In the notebook on the right, we see three representations of the Dirac delta f_1, f_2, f_3 when $\varepsilon \to 0$ or $N \to \infty$

In [14]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [25]:
```
x = np.linspace(-3,3,100)

N = 3*np.pi
epsilon = 0.2

dx = np.diff(x)[0]
```

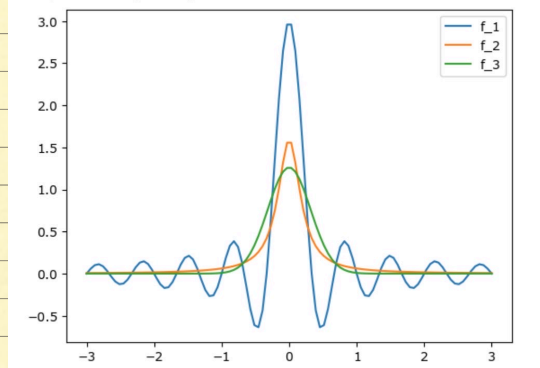
$$f_1(x) = \frac{1}{\pi}\frac{\sin(Nx)}{x} \qquad f_2(x) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2+t^2} \qquad f_3(x) = \frac{1}{\sqrt{\epsilon\pi}}e^{-x^2/\epsilon}$$

In [26]:
```
f1 = np.sin(N*x)/(np.pi*x)
np.sum(f1*dx)
```
Out[26]:
```
1.0218447934837154
```

In [28]:
```
f2 = (1/np.pi)*epsilon/(epsilon**2 + x**2)
np.sum(f2*dx)
```
Out[28]:
```
0.9580453308838972
```

In [29]:
```
f3 = 1/np.sqrt(epsilon*np.pi)*np.exp(-x**2/epsilon)
np.sum(f3*dx)
```
Out[29]:
```
0.9999999999999993
```

```
plt.plot(x,f1,label="f_1")
plt.plot(x,f2,label="f_2")
plt.plot(x,f3,label="f_3")
plt.legend()
```
Out[24]:
```
<matplotlib.legend.Legend at 0x16040fa10>
```



For all those functions, we see (numerically) that

$$\int_{-\infty}^{\infty} f_i(x)\,dx = 1 \quad , \qquad i=1,2,3$$

For a finite interval $x \in [-a, a]$, we can divide the integral as

$$\int_{-\infty}^{-a} dx\, f(x) + \int_{-a}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

and the first and third terms vanish for the asymptotic behavior (see plots on the right)
which justifies that in the general case $a < x < b$
the integral gives out

$$\int_a^b dx\,\delta(x-x_0) = \begin{cases} 1 & a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

Moreover, in the following case, we can write

$$\int_{-\infty}^{x} dt\,\delta(t-x_0) = \begin{cases} 1, & x_0 < x \\ 0, & \text{otherwise} \end{cases} = H(x-x_0)$$

which is the inverse one of the definitions of the Delta itself.

3. Dirac delta in several dimensions. We define an analogous distribution that goes to infinity at $\vec{r}=\vec{r}_0$ and zero everywhere else, with the integral in all space being one. In Cartesian coordinates, the integral definition can be written as

$$\int_{\text{all space}} \delta(\vec{r}-\vec{r}_0)\,dV = 1 \;;\; \int_{-\infty}^{\infty}\delta(x-x_0)\,dx \int_{-\infty}^{\infty}\delta(y-y_0)\,dy \int_{-\infty}^{\infty}\delta(z-z_0)\,dz = 1\cdot 1\cdot 1$$

But in other coordinate systems, one must be careful because of the Jacobian.

Problem: find the Dirac delta coordinate representation for $\delta(\vec{r}-\vec{r}_0)$ in 3D spherical coordinates.

$$\int_{\text{all space}} \delta(\vec{r}-\vec{r}_0)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \overset{!}{=} 1$$

This Dirac delta represents a point in sphace located at $\vec{r}_0$ (coords. r_0, θ_0, ϕ_0).

in order to fullfil the definition, let's postulate that the coordinate representation of the Delta is accompanied by factor functions

$$\delta(\vec{r}-\vec{r}_0) = a(r)\,b(\theta)\,c(\phi)\,\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)$$

$$\underbrace{\int_0^\infty a(r)\,r^2\,\delta(r-r_0)\,dr}_{\substack{\text{we need } a(r)\equiv\frac{1}{r^2} \\ \text{for this integral to be 1} \\ \text{(otherwise, it's } = a(r_0)r_0^2\text{)}}} \underbrace{\int_0^\pi b(\theta)\,\delta(\theta-\theta_0)\sin\theta\,d\theta}_{\substack{\text{we need } b(\theta)\equiv\frac{1}{\sin\theta} \\ \text{for this integral to be 1} \\ \text{(otherwise it's } = b(\theta_0)\sin\theta_0\text{)}}} \underbrace{\int_0^{2\pi} c(\phi)\,\delta(\phi-\phi_0)\,d\phi}_{\substack{\text{if } c(\phi)=1, \\ \text{this integral is already 1.}}} \overset{!}{=} 1$$

Problem: find the Dirac delta representation of an infinitesimally thin spherical shell located at $r = r_0$
In this case, the Dirac delta does not represent a point, but a spherical shell. This means that the delta must not depend on θ and ϕ (a spherical shell exists only at a given r , but it covers all points in θ and ϕ).

$$\delta_{\text{sph-shell}}(\vec{r}-\vec{r}_0) = a(r)\,\delta(r-r_0)$$

$$\int_{\text{all space}} \delta(\vec{r}-\vec{r}_0)\,dV \overset{!}{=} 1$$

$$\Rightarrow \int_0^\infty a(r)\,\delta(r-r_0)\,r^2\,dr \underbrace{\int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi}_{\substack{\text{this integral is} \\ =4\pi}} \overset{!}{=} 1$$

this means if $a(r)\equiv\frac{1}{4\pi r^2}$, the identity is satisfied $\Rightarrow$ $\delta_{\text{sph. shell}}(\vec{r}-\vec{r}_0) = \frac{1}{4\pi r^2}\,\delta(r-r_0)$

Problem: show that

$$\mathcal{F}\{\vec{\nabla} f(\vec{r})\}(\vec{k}) = i\vec{k}\,\mathcal{F}\{f(\vec{r})\}(\vec{k})$$

$$\int_{\text{all sp.}} \vec{\nabla} f(\vec{r})\,e^{-i\vec{k}\cdot\vec{r}}\,d^3x \overset{?}{=} i\vec{k}\int_{\text{all sp.}} f(\vec{r})\,e^{-i\vec{k}\cdot\vec{r}}\,d^3x$$

$\int u\,dv = uv - \int v\,du$

$i = \sqrt{-1}$

for each coordinate x_i,

$$\int \vec{e}^{\,i}\,\frac{\partial}{\partial x^i}(f(x^i))\,e^{-ik_jx^j}\,d^3x$$

by parts,

$$= \cancel{\vec{e}^{\,i} f(x^i)\,e^{-ik_jx^i}\Big|_{\text{boundaries}}}^{\,0} - \int_{\text{all space}} \vec{e}^{\,i} f(x^i)\,\frac{\partial}{\partial x^i}(e^{-ik_jx^j})\,d^3x$$

$$= -\int \hat{e}^i f(x^j)\, e^{-ik_j x^j} \cdot -k_j \delta_{ij}\, i\, d^3x$$

$$= -(-\hat{e}^i k_i i)\int f(x^j)\, e^{-ik_j x^j}\, d^3x$$

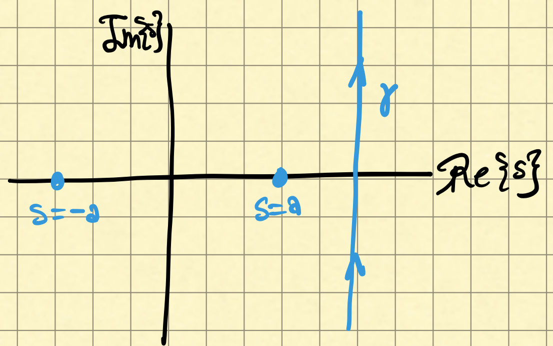
returning to vector notation,

$$= i\vec{k}\int_{\text{all sp.}} f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3x \quad ✓.$$

Problem: Compute the inverse Laplace transform of the function $f(s) = \frac{a}{s^2-a^2}$

using the Bromwich integral-.

$$F(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} f(s)\, ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{a\, e^{st}}{s^2-a^2}\, ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{a\, e^{st}}{(s+a)(s-a)}\, ds$$



$$= \frac{1}{2\pi i}\cdot 2\pi i \sum \text{residues}$$

$$= \frac{1}{\cancel{2\pi i}}\cdot \cancel{2\pi i}\left(\frac{e^{at}}{2} - \frac{e^{-at}}{2}\right) = \sinh at \,.$$

Problem: solve the partial differential equation

$$\nabla^2 \Phi = -4\pi \underbrace{Q\,\delta(\vec{r})}_{:=\rho(\vec{r})}$$

by Fourier transforms.

Note1: this corresponds to the electrostatic potential of a point charge located at $\vec{r}=\vec{0}$ because the enclosed charge is

$$Q = \int \rho(\vec{r})\, dV = \int Q\,\delta(\vec{r})\, dV = Q\int \frac{\delta(r)}{4\pi r^2}\, 4\pi r^2 dr = Q$$

Note2: This is the basis of the "Green function" method, that we discuss in this course.

~: transformed

$$\mathcal{F}\{\nabla^2\Phi\}(\vec{k}) = -4\pi Q\, \underbrace{\mathcal{F}\{\delta(\vec{r})\}(\vec{k})}_{1}$$

$$-k^2\,\tilde{\Phi} = -4\pi Q$$

$$\Rightarrow \tilde{\Phi} = \frac{4\pi Q}{k^2}$$

$$\Rightarrow \Phi = Q\,\overbrace{\frac{1}{(2\pi)^3}\, 4\pi}^{1/2\pi^2} \int \frac{1}{k^2}\, e^{i\vec{k}\cdot\vec{r}}\, d^3k$$

We use spherical coordinates and choose, without losing generality, $\vec{k}\cdot\vec{r} = kr\cos\theta$

$$\Phi(\vec{r}) = \frac{Q}{2\pi^2}\underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\infty} k^2 dk \int_{-1}^{1} d(\cos\theta)\, \frac{1}{k^2}\, e^{ikr\cos\theta}$$

$$= \frac{2Q}{\pi}\int_0^{\infty} \frac{\sin kr}{kr}\, dk = \frac{Q}{r} \quad ✓ \text{ (Gaussian units)}$$

Review problem: compute the Laplace transform of the function $f(t) = t^n$

$$F(s) = \int_0^\infty t^n e^{-st} dt = \text{parts} = \cancel{\frac{-t^n}{s} e^{-st}\Big|_0^\infty}^{0} + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

the situation "got better". Now we repeat n times until $t^0 = 1$.

$$= \frac{\overbrace{n(n-1)(n-2)\cdots 1}^{n!}}{\underbrace{s\cdot s\cdots s}_{n}} \underbrace{\int_0^\infty 1\cdot e^{-st} dt}_{\frac{-e^{-st}}{s}\Big|_0^\infty} = \frac{n!}{s^{n+1}}$$

Review problem: get the Laplace transform of $\cos t$ starting from the knowledge of the Laplace transform

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

We know that cosine is the derivative of sine. Then,

let $f(t) = \sin t \quad \circ\!\!-\!\!\bullet \quad F(s)$

$f'(t) = \cos t \quad \underset{\text{"maps to"}}{\circ\!\!-\!\!\bullet} \quad sF(s) - \underbrace{f(0)}_{\sin 0 = 0} = \frac{s}{s^2+1}$

Review problem: Solve the differential equation

$$y'' + 4y' + 4y = t^2 e^{-2t}$$

by Laplace transforms. (Boas) Use the initial conditions $y_0 = 0$, $y_0' = 0$

$$s^2 Y - \cancel{s y_0}^{0} - \cancel{y_0'}^{0} + 4sY - \cancel{4y_0}^{0} + 4Y = \frac{2}{(s+2)^3}$$

$$\Rightarrow Y = \frac{2}{(s+2)^5} \xrightarrow{\text{table}} y(t) = \frac{2t^4 e^{-2t}}{4!} = \frac{t^4 e^{-2t}}{12}.$$

table

$$\mathcal{L}\{t^k e^{-at}\} = \frac{k!}{(s+a)^{k+1}}$$

with $k > -1$

Problem: The following equation

$$a\,x(t) + \int_0^t K(t-\tau)\,x(\tau)\,d\tau = s(\tau) \quad , \; a \in \mathbb{R}$$

is called a Volterra integral equation of kernel K. The unknown function is $x(\tau)$ (therefore it's an integral equation). We can solve this kind of equation with the Laplace transform. Consider the particular Volterra integral equation

$$x(t) + \int_0^t (t-\tau)\,x(\tau)\,d\tau = \sin t$$

where the kernel is $K(t) = t$, $a = 1$ and $s(t) = \sin t$. Solve this equation by Laplace transforms.

$$\mathcal{L}\{x(t)\} + \mathcal{L}\Big\{\underbrace{\int_0^t (t-\tau)\,x(\tau)\,d\tau}_{\text{convolution}}\Big\} = \mathcal{L}\{\sin t\}$$

$$X(s) + K(s)\,X(s) = \frac{1}{s^2+1}$$

$$X(s) + \frac{1}{s^2} X(s) = \frac{1}{s^2+1}$$

$$\frac{(s^2+1)X(s)}{s^2} = \frac{1}{s^2+1} \Rightarrow X(s) = \frac{s^2}{s^2+1}$$

Using the tables

$$x(t) = \frac{\sin t + t\cos t}{2}$$

Context: we define the sine Fourier transform for an odd function as

$$g_-(p) = \mathcal{F}_s\{f\}(p) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, f_-(x) \sin(xp)$$

f_- is an odd function

Problem: compute the Fourier sine tranform for the function

$$f_-(x) = \begin{cases} x, & 0 \le x < 2\pi \\ 0, & \text{elsewhere} \end{cases}$$

$\int dv\, u = uv| - \int v\, du$

parts

Solution:

$$g_-(p) = \sqrt{\frac{2}{\pi}} \int_0^{2\pi} dx\, x \sin(xp) = \sqrt{\frac{2}{\pi}} \left. \frac{\sin xp - xp\cos xp}{p^2} \right|_0^{2\pi}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(2\pi p) - 2\pi p \cos(2\pi p)}{p^2}$$

Context: Analogous to the sine Fourier transform of an odd function, we define the cosine Fourier transform of an even function as

$$g_+(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, f_+(x) \cos(xp)$$

and

$$f_+(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty dp\, g_+(p) \cos(xp)$$

is the corresponding inverse transform. Notice that the Kernel and interval are identical in the direct and inverse cosine tranforms. (By the way, this is the case also for the inverse sine transform of the previous problem).

Problem: Consider the even function

$$f_+(x) = \delta(x)$$

$$\Rightarrow g_+(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, \delta(x) \cos(xp) = \sqrt{\frac{2}{\pi}} \cos(0 \cdot p) = \sqrt{\frac{2}{\pi}}$$

The inverse transform gives us

$$f_+(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty dp \sqrt{\frac{2}{\pi}} \cos(xp) \overset{!}{=} \delta(x) \Rightarrow \delta(x) = \frac{2}{\pi} \int_0^\infty dp \cos(xp)$$

This means we have found an *integral representation* of the delta function.