A 'pedestrian derivation' of the

Legendre polynomials

Consider the differential equation

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0 \qquad (1)$$

where $y:\mathbb{R}\to\mathbb{R}$, $y(x)$; $y' = \frac{dy}{dx}$; $y'' = \frac{d^2y}{dx^2}$,
$\ell \in \mathbb{R}$ but we want to know whether a solution exists $\forall \ell$.

This is a second-order linear ODE. We can solve this equation with a series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$, such that

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \dots + a_n x^n + \dots \qquad (2)$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \dots + n a_n x^{n-1} + \dots$$

$$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 \dots + n(n-1) a_n x^{n-2} + \dots$$

Example: $-x^2 y'' = -2a_2 x^2 + \dots$

Substituting (2) in each term of (1) and collecting the coefficients of each power,

we can find them with a change of the variable of the nth derivative of y.

	x^0	x^1	x^2	$\dots x^n \dots$
y''	$2a_2$	$6a_3$	$12a_4$	$(n+2)(n+1)a_{n+2}$
$-x^2y''$			$-2a_2$	$-n(n-1)a_n$
$-2xy'$		$-2a_1$	$-4a_2$	$-2na_n$
$+\ \ell(\ell+1)y$	$\ell(\ell+1)a_0$	$\ell(\ell+1)a_1$	$\ell(\ell+1)a_2$	$\ell(\ell+1)a_n$

according to (1), for each power

$= 0$

$$2a_2 + \ell(\ell+1)a_0 = 0 \Rightarrow a_2 = -\frac{\ell(\ell+1)}{2}a_0$$

$$6a_3 - 2a_1 + \ell(\ell+1)a_1 = 0 \Rightarrow a_3 = -\frac{(\ell-1)(\ell+2)}{\underbrace{6}_{=3!}}a_1$$

$$12a_4 - 2a_2 - 4a_2 + \ell(\ell+1)a_2 = 0 \Rightarrow a_4 = -\frac{(\ell-2)(\ell+3)}{12}a_2$$

$$= \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}a_0$$

For the n^{th} term, we find

$$a_{n+2} = -\frac{(\ell-n)(\ell+n+1)}{(n+2)(n+1)}\, a_n$$

← Recursion equation for the coefficients

The general solution of (1) is then

$$y = a_0\left[1 - \frac{\ell(\ell+1)}{2!}x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4 - \dots\right] + a_1\left[x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \dots\right]$$

constants of integration

Now, the question is: does this series converge?

- For x small, it does (higher order terms are even smaller). Use the ratio convergence criterion for series
- For $\ell=0$, $x=1$, for example

use recursion equation

$$y = a_0\left[1\right] + a_1\Big[\underbrace{1 + \tfrac{1}{3} + \tfrac{1}{5} + \dots}_{\text{harmonic series} \,\longrightarrow\, \text{divergent.}}\Big]$$

- For $\ell=1$, $x=1$, for example

$$y = a_0\left[1 - \frac{1\cdot 2}{2} + \frac{1\cdot \cancel{2}\cdot -1\cdot \cancel{4}}{\cancel{4}\cdot 3\cdot \cancel{2}} - \dots\right] + a_1 x$$

$$= a_0\Big[1 \underbrace{- 1 - \tfrac{1}{3} + \dots}_{\text{harmonic series} \,\longrightarrow\, \text{divergent}}\Big] + a_1 x$$

- For $\ell=2$, $x=1$, for example

$$y = a_0\left[1 - \frac{2\cdot 3}{2}\underset{\uparrow\, 1}{x^2} + \cancel{\frac{2\cdot 3\cdot 0\cdot 5}{4\cdot 3\cdot 2}} \to 0 + \dots\right] + a_1\Big[\underbrace{1 - \frac{1\cdot 4}{6} + \dots}_{\text{divergent}}\Big]$$

all other terms contain $(\ell-2)$

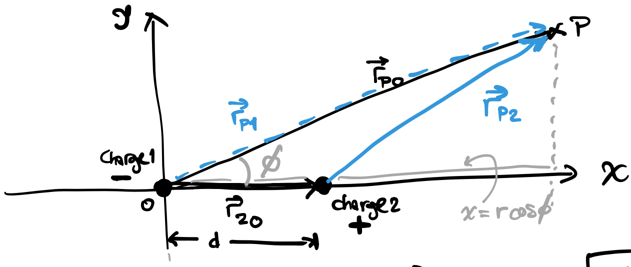
We are interested in solutions for $x \le 1$ such that $x = \cos\theta$, $0 \le \theta \le 2\pi$.
We obtain solutions for $\boxed{\ell \in \mathbb{N}_0}$ if the a_0, a_1 are switched off appropriately.
If we impose $y(1) = 1$, then the problematic coefficients vanish and we obtain the Legendre polynomials:

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_\ell(x)$ in general.

Legendre polynomials are orthogonal in the interval $x \in [1, 1]$.

Derivation of the Legendre polynomials from the expansion of a potential

Let's calculate the electrostatic potential of a dipole at a point P.



The charges are separated by a distance d. All the points are in the xy plane.

$$\vec{r}_{10} = \vec{0}$$
$$\vec{r}_{20} = d\,\vec{e}_x$$
$$\vec{r}_{P0} = x\vec{e}_x + y\vec{e}_y$$

then,

$$\vec{r}_{P1} = \vec{r}_{P0} \Rightarrow r_{P1} = \sqrt{x^2+y^2} = r$$
$$\vec{r}_{P2} = \vec{r}_{P0} - \vec{r}_{20} = (x-d)\vec{e}_x + y\,\vec{e}_y$$
$$\Rightarrow r_{P2} = \sqrt{(x-d)^2 + y^2}$$
$$= \sqrt{x^2 - 2xd + d^2 + y^2}$$
$$= \sqrt{r^2 + d^2 - 2rd\cos\phi}$$

The potential is then

$$V = V_1 + V_2 \qquad (1)$$
$$= \frac{-K}{r} + \frac{K}{\sqrt{r^2+d^2-2rd\cos\phi}}$$

where K absorbs the charges and constants.

Let's focus on $V_2 = \dfrac{K}{\sqrt{r^2+d^2-2rd\cos\phi}}$. Let's force r out of the square root:

$$V_2 = \frac{K}{r\sqrt{1-2\left(\frac{d}{r}\right)\cos\phi + \left(\frac{d}{r}\right)^2}}.$$

But then,

$$V = \frac{K}{r}\left[-1 + \underbrace{\left(1-2\left(\frac{d}{r}\right)\cos\phi + \left(\frac{d}{r}\right)^2\right)^{-1/2}}_{:=f}\right]$$

Changing variables to $\frac{d}{r} := h$ and $\xi = \cos\phi$, we obtain for the term f

$$f = (1 - 2h\,\xi + h^2)^{-1/2}$$

For $h \ll 1$ (large distances in comparison to the dipole separation), we can make a series expansion in h.

Taylor series expansion of $f(h) = (1 - 2h\xi + h^2)^{-1/2}$ around zero:

$$f(h) = f(0) + \frac{f'(0)}{1!}h + \frac{f''(0)}{2!}h^2 + \dots = \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(0)}{\ell!} h^\ell$$

where $' \equiv d/dh$

$$f(0) = 1$$

$$f'(0) = -\frac{1}{2}(1-2h\xi+h^2)^{-3/2} \cdot (-2\xi+2h)\Big|_{h=0} = -\frac{1}{2}\cdot -2\xi = \xi$$

$$f''(0) = -\frac{d}{dh}\left((1-2h\xi+h^2)^{-3/2}\cdot(h-\xi)\right)\Big|_{h=0}$$

$$= +\frac{3}{2}(1-2h\xi+h^2)^{-5/2}\cdot 2\cdot(h-\xi)^2 - (1-2h\xi+h^2)^{-3/2}\cdot 1\Big|_{h=0}$$

$$= 3(-\xi)^2 - 1 = 3\xi^2 - 1$$

etc. So, the first terms of the series are

$$f(h) = \underbrace{1}_{P_0(\xi)}\cdot 1 + \underbrace{\xi}_{P_1(\xi)} h + \underbrace{\frac{3\xi^2-1}{2}}_{P_2(\xi)} h^2 + \dots = \sum_{\ell=0}^{\infty} P_\ell(\xi)\, h^\ell .$$

The function $f(h) = (1-2h\xi+h^2)^{-1/2}$ is called *generating function*.

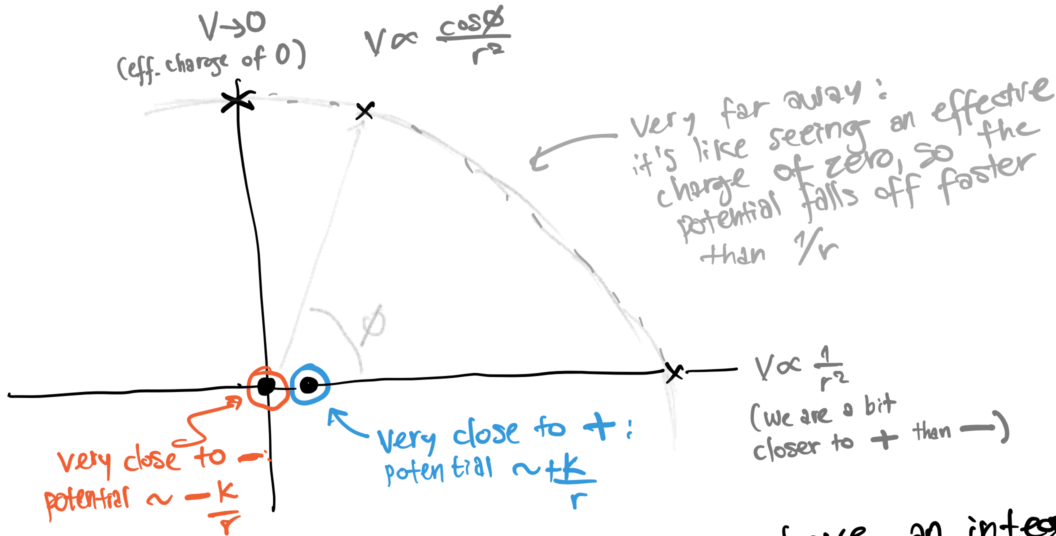
The potential is then

$$V = \frac{K}{r}\left[-1 + \sum_{\ell=0}^{\infty} P_\ell(\cos\phi)\left(\frac{d}{r}\right)^\ell\right] = \frac{K}{r}\left[\sum_{\ell=1}^{\infty} P_\ell(\cos\phi)\left(\frac{d}{r}\right)^\ell\right]$$

For $d \ll r$, the first term is

$$V = \frac{K}{r^2}\cdot \cos\phi \cdot d$$

The interpretation is simple: the potential falls off faster due to the reduced "effective charge" at large distances. See the figure in the next page



For a continuous distribution of mass/charge, we have an integral. The series decomposition of the potential is then called a *multipole expansion*.

Recurrence Formulae

For the generating function [that now we rename to $G(x,t)$ for convenience] we have

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$$

$G(x,t): \mathbb{R}\times\mathbb{R} \to \mathbb{R}$
$0 \le x \le 1$, $t \in \mathbb{R}$
$\ell \in \mathbb{N}_0$

Differentiating w.r.t. t,

$$\frac{\partial G}{\partial t} = \frac{-1}{2}\cdot(1-2xt+t^2)^{-3/2}\cdot(-2x+2t) = \frac{x-t}{(1-2xt+t^2)^{3/2}}$$

but we can write this as

$$(1-2xt+t^2)\frac{\partial G}{\partial t} = (x-t)\underbrace{\frac{1}{\sqrt{1-2xt+t^2}}}_{:=\, G(x,t)}.$$

Now, we substitute $\triangleright\ k \in \mathbb{N}_0$

$$G(x,t) = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$$

$$\begin{cases} \dfrac{\partial G(x,t)}{\partial t} = \dfrac{\partial}{\partial t}\displaystyle\sum_{k=0}^{\infty} t^k P_k(x) = \sum_{k=0}^{\infty} k\,t^{k-1}P_k(x) \\ -2xt\,\dfrac{\partial G}{\partial t} = -\displaystyle\sum_{k=0}^{\infty} 2xk\,t^k P_k \\ t^2\,\dfrac{\partial G}{\partial t} = \displaystyle\sum_{k=0}^{\infty} k\,t^{k+1}P_k \end{cases}$$

From now on, $P_k := P_k(x)$, $G := G(x,t)$

$$
+\left|\begin{array}{ll} -xG & = -\sum_{l=0}^{\infty} x t^l P_l \\ +tG & = +\sum_{l=0}^{\infty} t^{l+1} P_l \end{array}\right. \qquad =0
$$

$$
\sum_{k=0}^{\infty} k t^{k-1} P_k - \sum_{k=0}^{\infty} 2xk t^k P_k + \sum_{k=0}^{\infty} k t^{k+1} P_k - \sum_{l=0}^{\infty} x t^l P_l + \sum_{l=0}^{\infty} t^{l+1} P_l = 0
$$

We want to group all sums such that we can read out the coeff. of t^l.

$$
\sum_{k=0}^{\infty} k t^{k-1} P_k - \sum_{k=0}^{\infty} 2xk t^k P_k + \sum_{k=0}^{\infty} k t^{k+1} P_k - \sum_{l=0}^{\infty} x t^l P_l + \sum_{l=0}^{\infty} t^{l+1} P_l = 0
$$

$$
\underbrace{\sum_{k=0}^{\infty} (k+1) t^{k+1} P_k}_{\text{Let } l = k+1,\ k=0 \Rightarrow l=1} - \sum_{k=0}^{\infty} (2k+1) x t^k P_k + \underbrace{\sum_{k=0}^{\infty} k t^{k-1} P_k}_{\substack{1^{st} \text{ term} = 0 \Rightarrow k \text{ starts at } 1 \\ \text{let } l = k-1,\ k=1 \Rightarrow l=0}} = 0
$$

$\downarrow l := k$

$$
\sum_{l=1}^{\infty} l t^l P_{l-1} - \sum_{l=0}^{\infty} (2l+1) x t^l P_l + \sum_{l=0}^{\infty} (l+1) t^l P_{l+1} = 0
$$

Comparing coefficients <u>for $l \geq 1$</u>:

$$
\boxed{l P_{l-1} - (2l+1) x P_l + (l+1) P_{l+1} = 0}
$$

I Recurrence relation

<u>Example</u>: Knowing $P_0(x) = 1$, $P_1(x) = x$, generate $P_2(x)$ with the I recurrence rel.

$$
P_0 - 3x P_1 + 2P_2 = 0
$$

$$
\Rightarrow 1 - 3x^2 + 2P_2(x) = 0
$$

$$
\Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)
$$

Rodrigues' Formula

For the Legendre polynomials

The Legendre Polynomials can be generated from derivatives of $(x^2-1)^\ell$:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

Leibniz's rule for product differentiation

$$\frac{d^n}{dx^n}(f(x)g(x)) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^k g}{dx^k}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

That is,

$$\frac{d^n}{dx^n}(fg) = \underbrace{(D_f+D_g)^n}_{\text{binomial expansion}}(fg)$$

where $D_f(fg) = g\frac{df}{dx}$, etc.

(partial) Proof. Let's define $v = (x^2-1)^\ell$. Let's prove that $\frac{d^\ell v}{dx^\ell}$ is a solution of Legendre's equation.

Reminder:

$$(1-x^2)P_\ell''(x) - 2xP_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0$$

- $$(x^2-1)\frac{dv}{dx} = (x^2-1)\ell(x^2-1)^{\ell-1}\cdot 2x = \underbrace{(x^2-1)^\ell}_{v}\cdot 2x\ell = 2x\ell v$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2x\ell v$$

- Differentiating $\ell+1$ times (Leibniz's rule)

$f = dv/dx \qquad g = x^2-1$

$$(fg)^{(\ell+1)} = (D_f+D_g)^{(\ell+1)}(fg)$$

$$= D_f^{(\ell+1)}fg + \frac{(\ell+1)!}{1!\,\ell!}D_f^\ell D_g(fg) + \frac{(\ell+1)!}{2!(\ell-1)!}D_f^{\ell-1}D_g^2(fg)$$

but higher order deriv of. $g = 0$

$$\frac{d^{\ell+1}}{dx^{\ell+1}}\left[(x^2-1)\cdot\frac{dv}{dx}\right] = \frac{d^{\ell+1}}{dx^{\ell+1}}(2x\ell v)$$

$$(x^2-1)\frac{d^{\ell+2}v}{dx^{\ell+2}} + (\ell+1)(2x)\frac{d^{\ell+1}v}{dx^{\ell+1}} + \frac{(\ell+1)\ell}{2!}\cdot 2\cdot\frac{d^\ell v}{dx^\ell} = 2\ell x\frac{d^{\ell+1}v}{dx^{\ell+1}} + 2\ell(\ell+1)\frac{d^\ell v}{dx^\ell}$$

Simplifying,

$$(1-x^2)\underbrace{\left(\frac{d^\ell v}{dx^\ell}\right)''}_{P_\ell''(x)} - 2x\underbrace{\left(\frac{d^\ell v}{dx^\ell}\right)'}_{P_\ell'(x)} + \ell(\ell+1)\underbrace{\frac{d^\ell v}{dx^\ell}}_{P_\ell(x)} = 0$$

$$\Rightarrow (1-x^2)P_\ell''(x) - 2xP_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0$$

Which is the Legendre equation. We have proven that the eqn. is satisfied up to a factor. We won't prove the constant, but it's technically necessary to fully prove Rodrigues's formula.

Proof of the

Orthogonality of Legendre's polynomials

① orthogonality

We want to prove that $\langle P_\ell(x), P_m(x)\rangle = 0$ for $\ell \neq m$., $-1 \leq x \leq 1$.

that is, $\int_{-1}^{1} P_\ell(x) P_m(x)\, dx = 0, \quad \ell \neq m.$

▶ We write Legendre's equation as

$$\frac{d}{dx}\left[(1-x^2)P_\ell'\right] + \ell(\ell+1)P_\ell = 0$$

For the future: this is the Sturm-Liouville form of the Legendre diff. equation

Now we multiply $P_m(x)$:

$$P_m \frac{d}{dx}\left[(1-x^2)P_\ell'\right] + \ell(\ell+1) P_m P_\ell = 0 \qquad (1)$$

And do the same exchanging $m \leftrightarrow \ell$

$$P_\ell \frac{d}{dx}\left[(1-x^2)P_m'\right] + m(m+1) P_\ell P_m = 0 \qquad (2)$$

Now we do (1) − (2):

$$\underbrace{P_m \frac{d}{dx}\left[(1-x^2)P_\ell'\right] - P_\ell \frac{d}{dx}\left[(1-x^2)P_m'\right]}_{=\frac{d}{dt}\left[(1-x^2)(P_mP_\ell' - P_\ell P_m')\right]} + \left[\ell(\ell+1) - m(m+1)\right] P_m P_\ell = 0$$

$$\Rightarrow \frac{d}{dt}\left[(1-x^2)(P_mP_\ell' - P_\ell P_m')\right] + \left[\ell(\ell+1) - m(m+1)\right] P_m P_\ell = 0$$

Now let's form the inner product by integrating

$$\int_{-1}^{1} \frac{d}{dt}\left[(1-x^2)(P_mP_\ell' - P_\ell P_m')\right]dx + \left[\ell(\ell+1) - m(m+1)\right]\int_{-1}^{1} P_m P_\ell\, dx = 0$$

$$\cancel{(1-x^2)(P_mP_\ell' - P_\ell P_m')\Big|_{-1}^{1}} \to 0 \quad + \left[\ell(\ell+1) - m(m+1)\right] \langle P_m, P_\ell\rangle = 0$$

Then, since $\ell \neq m$,

$$\langle P_m, P_\ell\rangle \overset{!}{=} 0.$$

② Normalization constant (N^2)

We compute

$$\int_{-1}^{1} (P_\ell(x))^2 \, dx = N^2 = \langle P_\ell , P_\ell \rangle .$$

For this, we use the recursion formula

$$\ell P_\ell(x) = x P_\ell'(x) - P_{\ell-1}'(x)$$

Can be proven from the generating function ($\partial G/\partial x$) and combination with the other formula (from $\partial G/\partial t$) + algebra.

and the inner product

$$\ell \langle P_\ell, P_\ell \rangle = \langle x P_\ell , P_\ell' \rangle - \langle P_\ell , P_{\ell-1}' \rangle$$

integral by parts

$$\int_{-1}^{1} x P_\ell(x) P_\ell'(x) \, dx = \frac{x}{2} [P_\ell(x)]^2 \Big|_{-1}^{1} - \frac{1}{2} \int_{-1}^{1} [P_\ell(x)]^2 dx$$

$$= 1 - \frac{1}{2} \int_{-1}^{1} [P_\ell(x)]^2 dx$$

$$\int_{-1}^{1} P_\ell P_{\ell-1}' \, dx = 0$$

Explanation: $P_{\ell-1}'$ is a polynomial with a degree lower than ℓ. Because Legendre polynomials are a base of a vector space, can be written any polynomial of degree n as a linear combination of Legendre polynomials up to the degree n. So,

$$\langle P_\ell, P_{\ell-1}' \rangle = \sum_{k=0}^{n} c_k \langle P_\ell , P_n \rangle$$

But since $n < \ell$, the polynomials are orthogonal.

$$\Rightarrow \quad \ell \langle P_\ell , P_\ell \rangle = 1 - \frac{1}{2} \langle P_\ell , P_\ell \rangle$$

$$\Rightarrow (2\ell + 1) \langle P_\ell , P_\ell \rangle = 2$$

$$\Rightarrow \langle P_\ell , P_\ell \rangle = \frac{2}{2\ell+1}$$

⚠ The Legendre polynomials aren't orthonormal, so $\langle P_\ell , P_\ell \rangle \neq 1$.

In general, we write

$$\langle P_m(x) , P_\ell(x) \rangle = \int_{-1}^{1} P_m(x) P_\ell(x) \, dx = \frac{2}{2\ell+1} \delta_{m\ell} .$$