Notes on

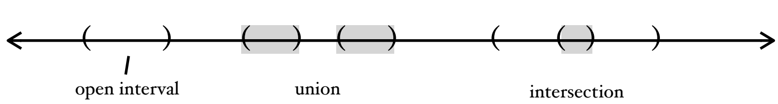
Mathematical Physics

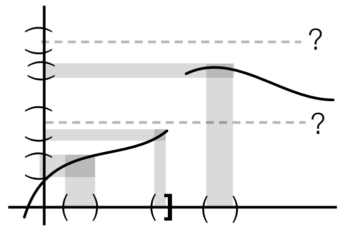
Differential geometry

Mathematical background: manifolds

Manifold $\mathscr{M}$: collection of **points**, smooth (meaning there are always more points between two given points), locally homeomorphic to $\mathbb{R}^n$ (a map can be built from any small section of $\mathscr{M}$ to a flat space).

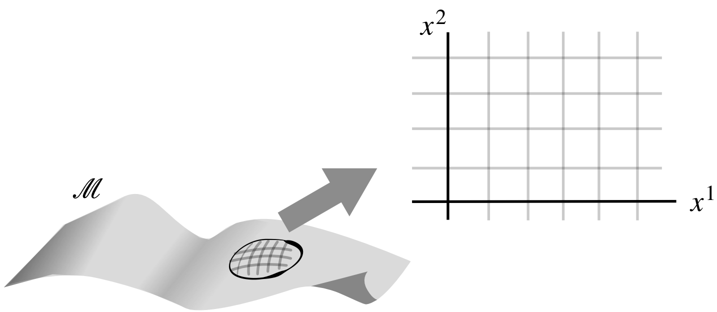
- To probe whether we have a manifold, use open intervals:





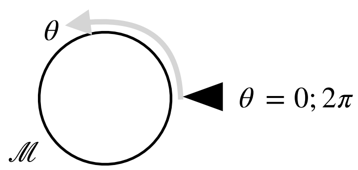
Notice that in a discontinuous (= non-smooth) function, there are open intervals in the vertical axis that correspond to (half-)closed intervals in the horizontal axis. In a manifold, we require smoothness.

Chart (coordinate patch): because a manifold is locally homeomorphic to $\mathbb{R}^n$, we can define a coordinate patch or chart as a local map from an open "interval" on the manifold to an Euclidean (locally flat) space. A collection of charts is called an **atlas**.



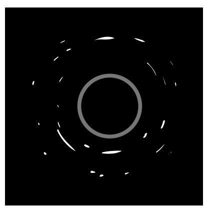
Examples of manifolds:

- Trivial: $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$
- Flat (Minkowski) spacetime: $\underbrace{\mathbb{R}^3}_{\text{space}} \times \underbrace{\mathbb{R}^1}_{\text{time}}$ (not equal to $\mathbb{R}^4$)
- Circle

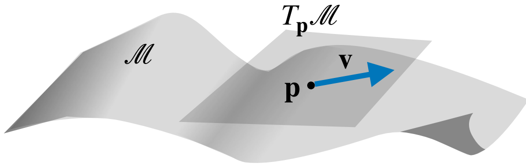


This manifold cannot be covered with only one coordinate patch! This is why we must be careful when programming and never take θ but $\theta \mod 2\pi$. The coordinate patch is usually taken as $0 \leq \theta < 2\pi$. A circle is locally smooth (similar to a straight line if we zoom in a lot).

- Spacetime around a compact object (neutron star, black hole)

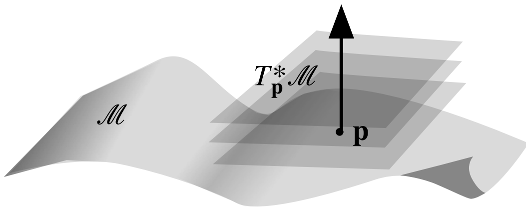


Mathematical background: vectors



Tangent space: One can define a tangent space of a manifold $\mathscr{M}$ at a point $\mathbf{p}$ ($T_\mathbf{p}\mathscr{M}$).

(Tangent) vectors: the elements $\mathbf{v}$ of the tangent space form a vector space. Because this is the tangent space, one can take as a basis as a tangent vector to the coordinate lines (u_i), that is, $\mathbf{d}_k := \frac{\partial \mathbf{r}}{\partial u^k}$ (derivative generates a tangent space). Therefore, a **vector v** has a *contravariant* representation as $\mathbf{a} = a^i \mathbf{d}_i$.



Dual space, covectors: one can define a *dual space* $T^*_\mathbf{p}\mathscr{M}$ by taking the perpendicular direction(s) to the tangent space (in the figure on the right, which depicts a 2D manifold, this results in a straight line but in general it can have more dimensions). This also forms a vector space, and the basis can be taken to be the perpendicular lines to the coordinate lines (u_i), that is, $\mathbf{d}^i := \nabla u^i$. The elements of $T^*_\mathbf{p}\mathscr{M}$ are usually called *covectors* or vectors represented in a *covariant form*, $\mathbf{a} = a_i \mathbf{d}^i$.

Tensors, tensor product: the tensor product $\otimes$ builds of a compound space using the tangent and dual spaces. For example, the tensor of components b^i_j is built as $\underline{b} = b^i{}_j \mathbf{d}_i \otimes \mathbf{d}^j$. (Einstein summation convention). The *rank* of a tensor is how many indices it has (a scalar has rank zero, vectors and covectors have rank 1, and the tensor $\underline{b}$ of the example has rank 2). The tensor product builds higher-ranked objects.

Metric: a way to define "distance" or interval ds within two points of a manifold separated by a length dx^i along each coordinate, can be provided by the metric, so that $ds^2 = g_{ij} dx^i dx^j$ ("generalization of Pythagoras's theorem"). The metric tensor is built as $\underline{g} = g_{ij} \mathbf{d}^i \otimes \mathbf{d}^j$.

Note that:

- One can convert between the covariant and contravariant representations of a vector **a** using the metric as $\mathbf{a} = a^i \mathbf{d}_i = a_i \mathbf{d}^i$ where $a_i = g_{ij} a^j$
- The previous point means that there is an inverse metric tensor with components g^{ij} such that $a^i = g^{ij} a_j$. Since they are the inverse of one another, $g^{ij} g_{jk} = \delta^i_k$.
- The dot product is defined as $\mathbf{a} \cdot \mathbf{b} = a^i b_i = a_i b^i$. This is more generally a **tensor contraction** and it is an operation that lowers the rank of a tensor.
- The bases are indeed the dual of one another: $\mathbf{d}^j \cdot \mathbf{d}_i = \nabla u^j \cdot \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial u^j}{\partial x_k} \frac{\partial x_k}{\partial u^i} = \frac{d u^j}{d u^i} = \delta^j_i$

Differential forms

One-forms: the basis of the covariant representation of a vector $\mathbf{d}^i$ can be taken as a differential $\mathrm{d}x^i$, so that we identify the objects $\mathbf{d}^i \equiv \mathrm{d}x^i \equiv \sigma$ (and other Greek letters) and call them a *one-form*.

Antisymmetric tensors: a tensor for which $T_{ij} = -T_{ji}$ is said to be antisymmetric. We can build an antisymmetric tensor of rank 2 from any tensor of the same rank by building the expression $\tilde{T}_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$.

Differential p-forms and the wedge product: instead of building an antisymmetric tensor via its components, we can also build an object called a *p-form,* which is an antisymmetric tensor of *covariant* rank p. To indicate their antisymmetry, we define the *wedge product* of two differential forms α, β as satisfying the property $\alpha \wedge \beta = -\beta \wedge \alpha$. Then, another way of building the basis for an antisymmetric tensor of rank 2 is by taking the basis $dx^i \otimes dx^j$ and instead building $dx^i \wedge dx^j$.

Vector space of differential forms: consider a vector field L of dimension n over a field (Körper) $\mathbb{R}$. Then, a vector in this space can be written as $\alpha = \sum_i a_i \sigma_i \in L$. The wedge product creates a product of the vector spaces, such that $\alpha \wedge \beta \in L \wedge L$. A *p-form* generated by the wedge product is then an element of the space denoted by $\bigwedge^p L$, that is, $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p \in \bigwedge^p L \equiv \underbrace{L \wedge L \wedge \cdots \wedge L}_{p \quad \text{times}}$, where $\bigwedge^0 L := \mathbb{R}$, $\bigwedge^1 L := L$. The wedge product generates a Grassman algebra.

Hodge star: given a p-form in a vector field of dimension n, it forms a *(n-p)-form* (complementary form built by the one-forms that are *not* in the original p-form). Careful: a) the wedge products are performed in the order of the coordinates (like the cross product) otherwise there should be a minus sign, and b) an extra minus sign can come from the metric (in Minkowski spacetime, this is the case if the form to be "hodged" contains dt).

Examples in Euclidean space: $\star\, 1 = dx \wedge dy \wedge dz$, $\star\, dx = dy \wedge dz$, $\star\, dz = dx \wedge dy$.
Examples in Minkowski spacetime: $\star\, 1 = dt \wedge dx \wedge dy \wedge dz$, $\star\, (dt \wedge dx) = -dy \wedge dz$.

Exterior derivative: Since we are using a differential as a one-form, we notice that the the total differential of a function $f(x, y, z)$ in $\mathbb{R}^3$, which we know is $df = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz$, takes a scalar function (i.e., a 0-form) and creates with it a 1 form. This operation then has the following properties:

- $d(\alpha + \beta) = d\alpha + d\beta$
- $d(a\alpha) = da \wedge \alpha + a\, d\alpha$, if a is a scalar and α is a p-form
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{p_\alpha} \alpha \wedge d\beta$
- For all forms α, $d(d\alpha) = 0$.
- In general, the exterior derivative is a map $d : \bigwedge^p \to \bigwedge^{p+1}$, $\omega \to d\omega$.

Integrals of p-forms: the integral over a p-form ω on the manifold $\mathscr{M}$ is defined as $\int_{\mathscr{M}} \omega = \int_U f(x^1, \ldots, x^p) dx^1 \cdots dx^p$, where $\omega = f dx^1 \wedge \cdots \wedge dx^p$ and U is a region of $\mathscr{M}$ with a chart where the coordinates x^i are defined. Using differential forms, the **Stokes theorem** becomes very simply $\int_{\partial S} \omega = \int_S d\omega$.