Ejercicios de funciones de Green

Problem: consider the differential equation

$$y'' + y = x$$

subject to the boundary conditions $y(0) = 0 \,, \quad y(\pi) = \pi$ Solve this equation by the Green function.

Solution: The Green function satisfies the differential equation

$$G''(x,t) + G(x,t) = \delta(x-t)$$

We start by solving the homogeneous differential equation
and applying the boundary conditions to each solution

$$y'' + y = 0 \quad , \quad x \in [0, \pi]$$

$$\Rightarrow \; y(x) = C_1 \cos x + C_2 \sin x$$

- $y(0) = 0 \Rightarrow C_1 = 0$
  $\Rightarrow y_1(x) = \sin x \quad (\text{we chose } C_2 = 1) \Rightarrow y_1'(x) = \cos x$
- $y(\pi) = \pi$
  $\Rightarrow y_2(x) = -\pi \cos x \quad (\text{we chose } C_1 = -\pi) \Rightarrow y_2'(x) = +\pi \sin x$

We form the Green function

$$\Rightarrow G(x,t) = \begin{cases} A\, y_1(x)\, y_2(t) \,, & x < t \\ A\, y_2(x)\, y_1(t) \,, & x > t \end{cases} = \begin{cases} -A \sin x \; \pi \cos t \,, & x < t \text{ i.e. } 0 \leq x < t \\ -A \; \pi \cos x \; \sin t \,, & x > t \text{ i.e. } t < x \leq \pi \end{cases}$$

$$A = \frac{1}{P(t)\left[y_2'(t)\, y_1(t) - y_1'(t)\, y_2(t)\right]}$$

$$= \frac{1}{1 \cdot \left[\pi \sin t \cdot \sin t + \cos t \; \pi \cos t\right]} = \frac{+1}{\pi}$$

$$\Rightarrow G(x,t) = \begin{cases} -\sin x \cos t \,, & x < t \\ -\cos x \sin t \,, & x > t \end{cases}$$

How this problem was generated:

$y = x + \sin x \qquad y(0) = 0$

$\qquad\qquad\qquad\quad\;\; y(\pi) = \pi$

$y' = 1 + \cos x$

$y'' = -\sin x$

$$\underbrace{-\sin x}_{y''} + \underbrace{x + \sin x}_{+\,y} \underbrace{- x}_{=\,x} = 0$$

With this, we can compute the solution for our particular source function.
Note: we have to be very careful in selecting which criterion of the Green function to integrate in t!

$$y(x) = \int_0^\pi G(x,t)\; t \; dt = \int_0^x \overset{t<x}{G_1(x,t)}\; t \, dt + \int_x^\pi \overset{t>x}{G_2(x,t)}\; t \; dt$$

$$= -\cos x \int_0^x \underbrace{t \sin t \, dt}_{\text{table (parts)}} \quad - \quad \sin x \int_x^{\pi} \underbrace{t \cos t \, dt}_{\text{table (parts)}}$$

$$\underbrace{\left[\sin t - t\cos t\right]_0^x}_{\sin x - x\cos x} \qquad\qquad \left[\cos t + t\sin t\right]_x^{\pi}$$

$$-1 - \cos x - x\sin x$$

$$= -\cos x(\sin x - x\cos x) - \sin x(-1 - \cos x - x\sin x)$$

$$= \cancel{-\cos x\sin x} + x\cos^2 x + \sin x + \cancel{\sin x\cos x} + x\sin^2 x$$

$$= \sin x + x$$

Problem #1

Problem: neutron diffusion in a nuclear reactor (or radiation transport in a star) can be studied as a diffusion problem. Let's consider the steady state diffusion equation. It can be shown that one can express it as

$$\nabla^2 \rho - \rho = -S(\vec{r})$$

where ρ is the number density of thermal neutrons and S is the "source density" of neutrons in a reactor.

a) Consider the case where the variables only depend on z in Cartesian coordinates. Write the Green function diff. equation.

In cartesian, 1D, z:

$$\frac{d^2 G(z,z')}{dz^2} - G(z,z') = -\delta(z-z')$$

b) Consider the case where the variables only depend on r in spherical coordinates. Write the Green function diff. equation and show that using the transformation $\overline{G} = rG$ one can transform the differential operator of the eqn to have the same form as the Cartesian case.

In spherical coord., only dep. on r,

$$\frac{1}{r}\frac{d^2 (r G(r,r'))}{dr^2} - G(r,r') = -\frac{\delta(r-r')}{4\pi r^2}$$

making $\overline{G}(r,r') = r G(r,r')$

we have

$$\frac{d^2}{dr^2}(\overline{G}(r,r')) - \overline{G}(r,r') = -\frac{\delta(r-r')}{4\pi r}$$

So, we have transformed the diff. operator to the same form as in the Cartesian 1D case.

Given info: one can transform the Green function from a Cartesian problem with $G(z,z') \to 0$ for $z \to \pm\infty$ into the Green function for a problem in spherical coordinates with the boundary condition $G(r,r') \to 0$ for $r \to +\infty$ using the expression

$$G_{sph}(r) = -\left[\frac{1}{2\pi z}\frac{\partial G_{cart}(z)}{\partial z}\right]_{z=r}$$

c) Find the Green function for the Cartesian case, i.e, the number density of thermal neutrons when the source is a plane located at $z=z'$ with vanishing boundaries at infinity.

$$\frac{d^2 G(z,z')}{dz^2} - G(z,z') = -\delta(z-z')$$

b.c. $G(z \to -\infty) = 0$
$G(z \to +\infty) = 0$

Homogeneus diff. eqn.:

$$\Rightarrow \left(\frac{d}{dz} + 1\right)\left(\frac{d}{dz} - 1\right) G_0(z) = 0$$

$\Rightarrow G_{01}(z) = e^{z}$

$G_{01}(z \to -\infty) = 0$ ✓

$\Rightarrow G_{02}(z) = e^{-z}$

$G_{02}(z \to +\infty) = 0$ ✓

$G_{01}'(z) = e^{z}$ $\qquad$ $G_{02}'(z) = -e^{-z}$

normaliz. factor

$$A = \frac{1}{1\left[e^{-z}e^{z} + e^{z}e^{-z}\right]} = \frac{1}{2}$$

$$\Rightarrow G(z, z') = \begin{cases} \frac{1}{2} e^{z} e^{-z'}, & z < z' \\ -\frac{1}{2} e^{-z} e^{z'}, & z > z' \end{cases}$$

d) Consider the case of a source plane located at the origin. Use the given transformation to find the Green function of a point source of thermal neutrons located at $r' = 0$. Hint: the radial coordinate is always $r > 0$.

$$G(z, z'=0) = \begin{cases} \frac{1}{2} e^{z}, & z < 0 \\ -\frac{1}{2} e^{-z}, & z > 0 \end{cases}$$

$$\frac{\partial G(z)}{\partial z} = \begin{cases} \frac{1}{2} e^{z}, & z < 0 \\ -\frac{1}{2} e^{-z}, & z > 0 \end{cases}$$

$$\Rightarrow G_{\text{point}}(r) = -\left[\frac{1}{2\pi z} \frac{\partial G_{>}(z)}{\partial z}\right]_{z=r}$$

$$\therefore G(r) = \frac{e^{-r}}{4\pi r}$$

e) Consider a plane source located at $z = z'$ and a boundary condition $G(z \to 0, z') = 0$.
Verify that the Green function
$G(z, z') = G_{pl}(z - z') - G_{pl}(z + z')$
where G_{pl} is the Green function for vanishing boundaries found in (c), satisfies the new boundary condition.
This is called the methdod of images. Hint: write the Green function in terms of absolute values.

The solution found can be written as

$$G_{pl}(z - z') = \begin{cases} \frac{1}{2} e^{+(z - z')}, & z < z' \\ \frac{1}{2} e^{-(z - z')}, & z > z' \end{cases} = \frac{e^{-|z - z'|}}{2}$$

therefore

$$G(z, z') = G_{pl}(z - z') - G_{pl}(z + z')$$

and for $z = 0$

$$\frac{e^{-|-z'|}}{2} - \frac{e^{-|z'|}}{2} = 0. \checkmark.$$

Problem: Solve with Green functions the boundary-value problem

boundary cond.:

$$-y'' = f(x) \quad , \quad y(0) = y(1) = 0$$

Homogen. eq.: $-y'' = 0 \Rightarrow y(x) = c_0 + c_1 x$

$y_1(x) = x$ (satisf. $y(0) = 0$)

$y_2(x) = 1 - x$ (satisf. $y(1) = 0$)

Sturm-Liouville: $p(x) = -1$, $s(x) = 0$, $w(x) = 1$

$\Rightarrow y_1'(x) = 1$, $y_2'(x) = -1$

$$\Rightarrow A = \frac{1}{-1\left[-1 \cdot x - 1(1-x)\right]} = 1$$

$$\Rightarrow G(x,t) = \begin{cases} x(1-t), & 0 \le x < t \quad (x < t) \\ t(1-x), & t < x \le 1 \quad (x > t) \end{cases}$$

$\therefore$ solution:

$$y(x) = \int_0^1 G(x,t)\, f(t)\, dt$$

Careful: here, we use the criterion that applies when integrating t, not x!
So, we write

$$G(x,t) = \begin{cases} x(1-t) & t > x \\ t(1-x) & t < x \end{cases}$$

Ex.: if $f(x) = \sin \pi x$,

$$y(x) = \int_0^1 G(x,t) \sin \pi t \, dt = (1-x)\int_0^x t \sin \pi t \, dt + x \int_x^1 (1-t) \sin \pi t \, dt$$

$$= \frac{1}{\pi^2} \sin \pi x$$

check of b. c.

$$y(0) = y(1) = 0$$

$$\frac{1}{\pi^2} \underbrace{\sin(\pi \cdot 0)}_{=0} = \frac{1}{\pi^2} \underbrace{\sin \pi}_{=0} \quad \checkmark$$

Problem: Solve with Green functions the ODE   Ex. 10.1.2 Arfken

$$\hat{\mathcal{L}} y = \frac{d^2 y}{dx^2} + y = f(x), \quad \underbrace{y(0) = 0, \; y'(0) = 0}_{\text{initial condition problem}}$$

Homog. $\left.\begin{matrix} y_1 = \sin x \\ y_2 = \cos x \end{matrix}\right\}$ none satisfy b. c. for $x = 0$, so $y = 0$ (trivial)

$$G(x,t) = 0, \quad x < t.$$

For $x > t$: (no b. c.)

$$G(x,t) = C_1(t) \sin x + C_2(t) \cos x, \quad x > t$$

continuity cond.:

$$G(t_-, t) = G(t_+, t) \Rightarrow 0 = C_1(t) \sin t + C_2(t) \cos t$$

$$\frac{\partial G}{\partial x}(t_+, t) - \frac{\partial G}{\partial x}(t_-, t) = \frac{1}{P(t)} = 1 \Rightarrow C_1(t) \cos t - C_2(t) \sin t - 0 = 1$$

3D

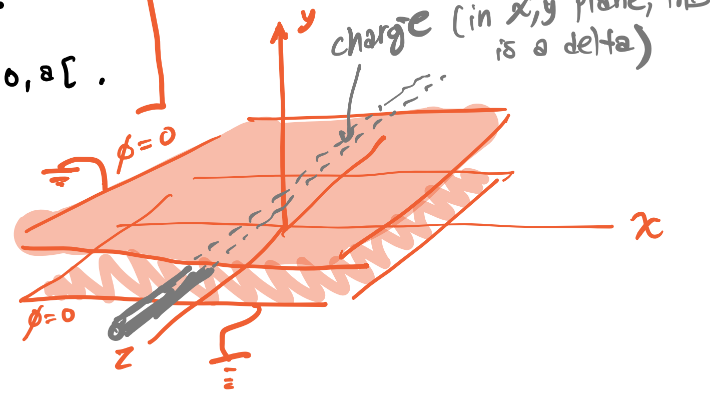
Problem: solve the Poisson equation with Green functions in Cart. coordinates

We'll follow Kraut "Fundamentals of math. physics" section 7-9 (p. 323) and 7-11 (p. 343).

7-9) key points:

$$\nabla^2 G = -\delta(x-x')\,\delta(y-y') \quad , \quad G = G(x,x';y,y')$$

$$\left.\begin{array}{ll} G(x,0)=0 & x\in\mathbb{R} \\ G(x,a)=0 & \\ \lim\limits_{|x|\to\infty} G(x,y)=0 & y\in\,]0,a[\,. \end{array}\right]\ \text{boundary conditions}$$



Homogeneous: $\nabla^2 G = 0 \qquad G = X(x)\,Y(y)$

const. of sep.

$$\begin{cases} X''(x) \mp k^2 X(x) = 0 \\ Y''(y) \pm k^2 Y(y) = 0 \end{cases}$$

$+k^2$:

$$G = XY = \big[\underbrace{D(k)e^{kx} + E(k)e^{-kx}}_{X(x)}\big]\big[A(k)\sin \underset{n\pi y/a}{ky} + \overset{0}{\cancel{B(k)}}\cos ky\big]$$

Bound. cond. $k = n\pi/a \Rightarrow B = 0$

$-k^2$:

$$G = XY = \big[D(k)\cos ky + E(k)\sin kx\big]\big[A(k)\sinh ky + B(k)\cosh ky\big]$$

$$G = \sum_{n=1}^{\infty} X_n(x)\sin\frac{n\pi y}{a}$$

bound. cond. $\lim_{|x|\to\infty} X_n(x) = 0$

$$\nabla^2 G = -\delta(x-x')\delta(y-y')$$

↓expand δ in eigenfunctions of y

$$-\delta(x-x')\delta(y-y') = -\sum_{n=1}^{\infty} \delta(x-x') A_n \sin\frac{n\pi y}{a}$$

coef.

$$A_n = \frac{2}{a}\sin\frac{n\pi y'}{a}$$

$$= -\frac{2}{a}\sum_{n=1}^{\infty} \delta(x-x') \sin\left(\frac{n\pi y'}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{d^2X_n}{dx^2} - \left(\frac{n\pi}{a}\right)^2 X_n\right]\sin\frac{n\pi y}{a} = -\frac{2}{a}\sum_{n=1}^{\infty}\delta(x-x')\sin\frac{n\pi y'}{a}\sin\frac{n\pi y}{a}$$

$$\Rightarrow \frac{d^2X_n}{dx^2} - \left(\frac{n\pi}{a}\right)^2 X_n = -\frac{2}{a}\sin\frac{n\pi y'}{a}\,\delta(x-x')$$

this is a 1D Green function problem. particular soln. with wronskians can be written as

$$X_n(x) = \frac{1}{n\pi}\sin\frac{n\pi y'}{a}\, e^{-n\pi x_>/a}\, e^{n\pi x_</a}$$

where $x_>$ = greater of (x, x')
$x_<$ = lesser of (x, x')

$$\boxed{\therefore\ G(x,x';y,y') = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi y'}{a}\sin\frac{n\pi y}{a}\, e^{-n\pi x_>/a}\, e^{n\pi x_</a}}$$

$$G = \int_{-\infty}^{\infty}\left[D(k)\cos kx + E(k)\sin kx\right]\left[A(k)\sinh ky + B(k)\cosh ky\right]dk$$

here we can't choose $k = n\pi/a$ to satisfy b.c.

We use the integral representation (based on a cosine Fourier transform):

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos k(x-x')\,dk$$

$$-\delta(x-x')\delta(y-y') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(y-y')\cos k(x-x')\,dk$$

we choose D, E such that

$$G = \int_{-\infty}^{\infty} \underbrace{Y(k,y)}_{A(k)\sinh ky + B(k)\cosh ky}\cos k(x-x')\,dk$$

now, diff. eqn. again:

$$\int_{-\infty}^{\infty}\left(\frac{d^2Y}{dy^2}-k^2Y\right)\cos k(x-x')\,dk = \frac{-1}{2\pi}\int_{-\infty}^{\infty}\delta(y-y')\cos k(x-x')\,dk$$

$$\Rightarrow \frac{d^2Y}{dy^2}-k^2Y = -\frac{1}{2\pi}\delta(y-y')$$

again, this is a 1D ODE, b.c. $Y(0)=0$, $Y(a)=0$

solution with 1D Green function

$$Y(k,y) = \frac{\sinh k(a-y_>)\sinh ky_<}{2\pi k\sinh ka}$$

$$\boxed{G(x,x';y,y') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh k(a-y_>)\sinh ky_<}{k\sinh ka}\cos k(x-x')\,dk}$$

Problem: Find the Green function for Harmonic oscillator using Laplace transform

$$y''+\omega^2 y = f(t) \qquad y_0 = y_0' = 0$$

$$f(t) = \int_0^{\infty} f(t')\,\delta(t'-t)\,dt' \quad ; \quad y(t) = \int_0^{\infty} G(t,t')\,f(t')\,dt'$$

$$\frac{d^2}{dt^2}G(t,t') + \omega^2 G(t,t') = \delta(t'-t)$$

$$\mathcal{L}\left\{\frac{d^2}{dt^2}G(t,t')\right\} + \omega^2\mathcal{L}\{G(t,t')\} = \mathcal{L}(\delta(t'-t))$$

$$s^2\mathcal{L}\{G(t,t')\} - s\cancel{G(0,t')}^{\,0} - \cancel{\partial_t G(t,t')|_{t=0}}^{\,0} + \omega^2\mathcal{L}\{G(t,t')\} = e^{-st'}$$

$$\mathcal{L}\{G(t,t')\} = \frac{e^{-st'}}{s^2+\omega^2}$$

$$\Rightarrow G(t,t') = \frac{\sin(\omega(t-t'))}{\omega}H(t-t')$$