Generalized Legendre differential equation

$$\frac{d}{dx}\left((1-x^2)\, y'(x)\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right) y(x) = 0 \quad , \qquad m^2 \le \ell^2$$

For m = 0, we recover the Legendre differential equation.

For $m \neq 0$, we can use the Ansatz

$$y(x) = (1-x^2)^{m/2}\, u(x)$$

to obtain the equation

$$(1-x^2)\, u''(x) - 2(m+1)\, x\, u'(x) + \left[\ell(\ell+1) - m(m+1)\right] u(x) = 0 \qquad (*)$$

If we take the derivative of this equation w.r.t. x,

$$(1-x^2)\left[u'(x)\right]'' - 2(m+2)\, x \left[u'(x)\right]' + \left[\ell(\ell+1) - m(m+1)(m+2)\right]\left[u'(x)\right] = 0$$

which is the same as $(*)$ if one makes the substitution $m \to m+1$, $u(x) \to u'(x)$

If we continue to do derivatives, we can see that the pattern

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \qquad 0 \le m \le \ell$$

(Asociate Legendre functions) is a solution of the generalized Legendre differential equation.
If $m < 0$, then we have

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

The derivative of the Legendre polynomials is called the associate Legendre polynomials, and if it is multiplied by the factor of $(1-x^2)^{m/2}$, then it is the asociated Legendre function.

Simple example: find the Rodrigues' formula for the assoc. Legendre functions.
Solution:

$$P_\ell^m = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m}\left(\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right)$$

$$= \frac{(-1)^m (1-x^2)^{m/2}}{2^\ell \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \quad , \qquad \ell \geq 0, \ |m| \le \ell, \ m \in \mathbb{Z}, \ \ell \in \mathbb{N}_0$$

Legendre functions of the second kind

The Legendre ODE is of second order. This means there must be two linearly independent solutions. We can find a second solution of a general second-order ODE

$$y'' + F(x)\, y' + G(x)\, y = 0$$

by computing

$$y_2(x) = y_1(x) \int \frac{e^{-\int F(x_1)\, dx_1}}{[y_1(x_2)]^2}\, dx_2 \, .$$

If we apply this to the Legendre equation

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} - \frac{\ell(\ell+1)}{1-x^2}y = 0$$

we obtain, for example for $\ell=0$:

$$Q_0(x) = \dots = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

This is valid in $|x|<1$ and diverges in $|x|=1$. There are also Associated Legendre Functions of the Second Kind.
All the Legendre functions of the second kind satisfy the same recurrence relations as their counterparts of the first kind.

Example: find the normalization of the Legendre polybnomials of the first kind using the generating function.
Solution:

If we multiply the generating function by itself, we obtain (look up "rules for multiplying two series together")

$$\Phi^2(x,t) = \left[\sum_{\ell=0}^{\infty} t^\ell P_\ell(x)\right]^2 = \sum_{\ell,m} h^{\ell+m} P_\ell(x) P_m(x)$$

We integrate on both sides

$$\int_{-1}^{1}\Phi^2(x,t)\,dx = \sum_{\ell,m} t^{\ell+m}\int_{-1}^{1} P_\ell(x)P_m(x)\,dx = \sum_{\ell=0}^{\infty} t^{2\ell}\int_{-1}^{1}dx\,[P_\ell(x)]^2$$

Let's manipulate the l.h.s:

$$\int_{-1}^{1}dx\,\Phi^2(x,t) = \int dx\,\frac{1}{1-2xt+t^2} = -\frac{1}{2t}\ln\frac{1-2t+t^2}{1+2t+t^2} = \frac{1}{t}\ln\left|\frac{1+t}{1-t}\right| \overset{\text{power series, } |t|<1}{=} 2\sum_{\ell=0}^{\infty}\frac{t^{2\ell}}{2\ell+1}$$

Finally, comparing against the r.hs., we conclude that

$$\int_{-1}^{1}dx\,[P_\ell(x)]^2 = \frac{2}{2\ell+1}$$

Gamma function (introduction/review)

The function $\Gamma(p)$ is defined recursively as

$$\Gamma(p+1) = p\,\Gamma(p)$$

In case p is an integer, then we obtain the **factorial function**:

$$\Gamma(p) = (p-1)!$$

So, the Gamma function is a generalization of the factorial function for negative and non-integer arguments.
For example, one can show that

$$\Gamma(1/2) = \sqrt{\pi}$$

and that $\Gamma(-0.4)$ (non-integer negative argument) exists, but $\Gamma(-3)$ (negative integer argument) diverges.

Bessel's equation

When we solve the ODE

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

by the series solution method, we find an explicit expression for the Bessel functions of the 1st kind of order p:

$$y(x) \equiv J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

for integer $p > 0$. For integer $p < 0$, the $\Gamma(n-p+1)$ function diverges if $n \leq p-1$
Then, the first terms of the series are going to be zero $(1/\infty \to 0)$
until the series starts as in the positive case. One can show that

$$J_{-p}(x) = (-1)^p J_p(x)$$

so that the solution with positive and negative integer p are not linearly independent. We need to find another solution

Bessel functions of the second kind

For integer p, the function

$$Y_p(x) = \lim_{\nu \to p} \frac{\cos(\pi\nu)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)} \qquad (\text{also} := N_p(x))$$

provides a linearly independent solution to the Bessel ODE. (Neumann or Weber function)
In summary, the full solution is

$$y(x) = A J_p(x) + B Y_p(x), \quad p \in \mathbb{Z}$$

Note that if p is not an integer, we can use the negative p as a linearly independent solution and in that case, the general solution is

$$y(x) = A J_p(x) + B J_{-p}(x), \qquad p \text{ non-integer}$$

Bessel ODE as a Sturm-Liouville problem

The differential operator of the Bessel ODE can be written as

$$x^2 y'' + x y' + (x^2 - p^2) y = 0 \quad \Rightarrow \quad x \frac{d}{dx}\left(x \frac{d}{dx}\right) y + (x^2 - p^2) y = 0$$

dividing by x,

$$\frac{d}{dx}\left(x\frac{d}{dx}\right)y - \frac{p^2}{x}y + 1\cdot x\cdot y = 0$$

now we can identify

$\lambda = 1$

$w(x) = x$

$p_{SL}(x) = x$ (Sturm-Liouville)

$s(x) = -p^2/x$ (p: order of the function)

Note: it is customary to make the change of variables $x := \kappa\rho \Rightarrow \rho = x/\kappa \Rightarrow \frac{d}{dx} = \frac{d\rho}{dx}\cdot\frac{d}{d\rho} = \frac{1}{\kappa}\frac{d}{d\rho} \Rightarrow \frac{d^2}{dx^2} = \frac{1}{\kappa^2}\frac{d^2}{d\rho^2}$
with which it is easy to show that the eigenvalue becomes $\lambda = \kappa^2$ instead. (This will be very useful when working with partial differential equations). See Generalized Bessel ODE below with $b=\kappa$, $c=1$, $a=0$.

Orthogonality and normalization

The Bessel functions build an orthogonal set in the interval $[0, 1]$
as long as the Dirichlet boundary condition is imposed, that is

$$J_p(a) = J_p(b) = 0 \qquad \text{(so } a \text{ and } b \text{ are zeros of the Bessel Function)}$$

It is important to note that there are infinitely many zeros of the Bessel function, and so, we have to specify which zero to use for the orthogonality and normalization.

The normalization of the Bessel functions depends on the zero chosen:

$$\int_0^1 dx\; x\, J_p(ax)\, J_p(bx) = \begin{cases} 0, & a\neq b \\ \underbrace{\tfrac{1}{2}[J_p'(a)]^2 = \tfrac{1}{2}[J_{p+1}(a)]^2 = \tfrac{1}{2}[J_{p-1}(a)]^2}_{\text{the equivalence of these equations can be shown with recursion relations (obtained from deriv. of generating function)}}, & a=b \end{cases}$$

⚠ rescaling done here!

There are n zeros. If we order the zeros of the Bessel function as $a_1, a_2, \dots, a_n$
then we say that the functions $J_p(a_n x)$
are orthogonal in the interval $[0, 1]$ with the weight function $w(x) = x$

...Or that the functions $\sqrt{x}\, J_p(a_n x)$
build an orthogonal set in the same interval with a weight function $w(x) = 1$

Problem: show the normalization of the Bessel functions. Use the change of variables $\rho \to \kappa x \Rightarrow$

Sturm-Liouville, rescaled

$$\begin{cases} p_{SL}(\kappa_n\rho) = \kappa_n\rho \\ s(\kappa_n\rho) = -p^2/(\kappa_n\rho) \\ \lambda = \kappa_n^2 \\ \text{interval: } [0, a] \\ w(\kappa_n\rho) = \rho \end{cases}$$

Note: this is $[0,1]$ if the variable of integration is $(\kappa_n\rho) = x$, but if it is just ρ, then it becomes $[0, a]$ (rescaling)
∴ without rescaling $J_p(a) = 0$
∴ with rescaling $J_p(\kappa a) = 0$.

From the PDF "Sturm-Liouville", § "Orthogonality... with weigh function" we have

$$\underbrace{p_{SL}(\kappa\rho)\left[J_p(\kappa_m\rho)\,J_p'(\kappa_n\rho) - J_p'(\kappa_m\rho)\,J_p(\kappa_n\rho)\right]_0^a}_{\text{evaluating at } \rho=a \text{ and } \rho=0,} = (\kappa_m^2 - \kappa_n^2)\int_0^a d\rho\; \rho\, J_p(\kappa_m\rho)\, J_p(\kappa_n\rho)$$

$$\frac{a\left[\kappa_n J_p(\kappa_m a)\, J_p'(\kappa_n a) - \kappa_m J_p'(\kappa_m a)\, J_p(\kappa_n a)\right]}{\kappa_m^2 - \kappa_n^2} = \int_0^a d\rho\; \rho\, J_p(\kappa_m\rho)\, J_p(\kappa_n\rho)$$

If we take the limit as $\kappa_n \to \kappa_m$, we find $0/0$. We use L'Hôpital

$$\int_0^a \rho\left[J_p(\kappa_m\rho)\right]^2 d\rho = \lim_{\kappa_n\to\kappa_m} \frac{a\left[J_p(\kappa_m a)\frac{d}{d\kappa_n}\kappa_n J_p'(\kappa_n a) - \kappa_m J_p'(\kappa_m a)\frac{d}{d\kappa_n}J_p(\kappa_n a)\right]}{\frac{d}{d\kappa_n}(\kappa_m^2 - \kappa_n^2)}$$

and now set $J_p(\kappa_m a) = 0$ (boundary) to find

$$= -a^2\kappa_m\left[J_p'(\kappa_m a)\right]^2/(-2\kappa_m) = \frac{a^2}{2}\left[J_p'(c_{pm})\right]^2$$

by definition, zeros of J_p

Simple problem: suppose that, for a problem defined in $0 \leq r \leq a$, we require $J_0(\kappa a) = 0$.
with k being a constant to be determined. What are the acceptable values of k ? $\kappa a = c_{0m}$, $m = 1, 2, \dots$

zeros of Bessel 0th order you can compute it with Python

$$\Rightarrow \kappa = c_{0m}/a$$

Write the Bessel function that satisfies the boundary condition:

$$J_0\left(\frac{c_{0m}}{a}\cdot r\right)$$

Generalized Bessel ODE

Bessel functions form a family of solutions of the general ODE

$$y'' + \frac{1-2a}{x} y' + \left[(b c x^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2} \right] y = 0$$

with constant a, b, c, p

The solution of this equation is $y = x^a Z_p(b x^c)$

Z_p is J, N or a combination of them.

Further Bessel functions

$$\underbrace{H_\nu^{(1)}(x) := J_\nu(x) + iY_\nu(x), \quad H_\nu^{(2)}(x) := J_\nu(x) - iY_\nu(x)}_{\text{used because of analogy to } e^{\pm i\theta} = \cos\theta \pm i\sin\theta \text{ (traveling waves)}} : \text{Hankel functions}$$

- **Modified** Bessel ODE

different w.r.t. Bessel

$$x^2 y'' + xy' - (x^2 + p^2) y = 0$$

Solution: $y = Z_p(ix)$

$$I_p(x) = i^{-p} J_p(ix)$$

$$K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix)$$

- Spherical Bessel ODE

$$p = (2n+1)/2 = n + 1/2 \ , \quad n \in \mathbb{Z}$$

$$x^2 y'' + 2xy' + [x^2 - n(n+1)]y = 0$$

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x)$$

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+1/2}(x) = j_n(x) + i\, y_n(x)$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+1/2}(x) = j_n(x) - i\, y_n(x)$$