Ejercicio de repaso: considere un punto p en una variedad M. El plano tangente en p está dado por la ecuación

$$\vec{r} = u^1\left(1, \frac{2}{3}\right) + u^2\left(1, -\frac{2}{3}\right)$$

con u^1, u^2 parámetros del plano, que serán nuestras coordenadas. Calcule:

a) Los vectores base tangentes (base contravariente)

$$\vec{a}_1 = \frac{\partial \vec{r}}{\partial u^1} = \left(1, \frac{2}{3}\right)$$

$$\vec{a}_2 = \frac{\partial \vec{r}}{\partial u^2} = \left(1, -\frac{2}{3}\right)$$

b) Los covectores base (base covariante) $\vec{a}^1 = \vec{\nabla} u^1$, $\vec{a}^2 = \vec{\nabla} u^2$. Ocupamos $u^1(x,y)$, $u^2(x,y)$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = u^1\begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} + u^2\begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix} \xrightarrow{\cdots} \begin{cases} u^1 = (2x+3y)/4 \\ u^2 = (2x-3y)/4 \end{cases}$$

$$\vec{a}^1 = \vec{\nabla} u^1 = \left(\frac{1}{2}, \frac{3}{4}\right)$$

$$\vec{a}^2 = \vec{\nabla} u^2 = \left(\frac{1}{2}, \frac{-3}{4}\right)$$

c) El vector $\vec{a} = a^1\vec{a}_1 + a^2\vec{a}_2 = \vec{a}_1 + \frac{4}{3}\vec{a}_2$ en la base covariante.

Resolvemos el sist. de ec: $\vec{a}_1 + \frac{4}{3}\vec{a}_2 = a_1\vec{a}^1 + a_2\vec{a}^2$, insertando los vectores numéricos

$$\xrightarrow{\cdots} \begin{cases} a_1 = 59/27 \\ a_2 = 67/27 \end{cases}$$

d) La norma

$$|\vec{a}|^2 = a^i \underset{\text{suma}}{a_i} = \cdots = \frac{445}{81}$$

e) El tensor métrico

$$(g_{ij}) = \begin{pmatrix} \vec{a}_1\cdot\vec{a}_1 & \vec{a}_1\cdot\vec{a}_2 \\ \vec{a}_2\cdot\vec{a}_1 & \vec{a}_2\cdot\vec{a}_2 \end{pmatrix} = \begin{pmatrix} \frac{13}{9} & \frac{5}{9} \\ \frac{5}{9} & \frac{13}{9} \end{pmatrix}$$

veamos: $a_i = g_{ij}a^j$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{13}{9} & \frac{5}{9} \\ \frac{5}{9} & \frac{13}{9} \end{pmatrix}\begin{pmatrix} 1 \\ 4/3 \end{pmatrix} = \begin{pmatrix} \frac{13}{9} + \frac{5}{9}\cdot\frac{4}{3} \\ \frac{5}{9} + \frac{13}{9}\cdot\frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{59}{27} \\ \frac{67}{27} \end{pmatrix} \checkmark$$

Problema: sean las siguientes 1 formas en el espacio $\Lambda^2 L$:

$$a = a_1\sigma_1 + a_2\sigma_2 \qquad b = b_1\sigma_1 + b_2\sigma_2$$

Demuestre que al hacer $a\wedge b$ queda el determinante de la matriz $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$$a\wedge b = (a_1\sigma_1 + a_2\sigma_2)\wedge(b_1\sigma_1 + b_2\sigma_2)$$

$$= \underbrace{a_1 b_1 \sigma_1\wedge\sigma_1}_{0} + a_1 b_2\sigma_1\wedge\sigma_2 + a_2 b_1 \sigma_2\wedge\sigma_1 + \underbrace{a_2 b_2 \sigma_2\wedge\sigma_2}_{0}$$

$$= a_1 b_2\,\sigma_1\wedge\sigma_2 + a_2 b_1 \underbrace{\sigma_2\wedge\sigma_1}_{-}$$

$$= (a_1 b_2 - a_2 b_1)\,\sigma_1\wedge\sigma_2$$

Problema: Calcule el producto cuña entre un elemento $\alpha\in\Lambda^1 L$ y otro $\beta\in\Lambda^2 L$
y demuestre que se obtiene un elemento de $\Lambda^3 L$ cuya componente es el producto punto $\vec{a}\cdot\vec{b}$
donde las componentes de α son a_i, y las de β son b_i.

$$\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$$
$$\beta = b_1 dx_2\wedge dx_3 + b_2 dx_3\wedge dx_1 + b_3 dx_1\wedge dx_2$$

- - - - $= 0$

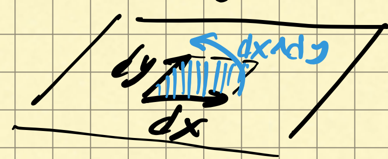
———— $\neq 0$

$$\alpha\wedge\beta = (a_1 dx_1 + a_2 dx_2 + a_3 dx_3)\wedge(b_1 dx_2\wedge dx_3 + b_2 dx_3\wedge dx_1 + b_3 dx_1\wedge dx_2)$$

$$= a_1 b_1 dx_1\wedge dx_2\wedge dx_3 + a_2 b_2 dx_2\wedge dx_3\wedge dx_1 + a_3 b_3 dx_3\wedge dx_1\wedge dx_2$$

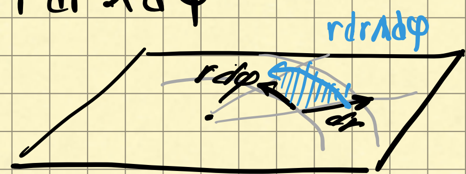
$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)\, dx_1\wedge dx_2\wedge dx_3 .$$

Problema: utilizando formas diferenciales, transforme el diferencial de área en coordenadas cartesianas $dx\,dy$ a un diferencial de área en coordenadas polares



$$x = r\cos\varphi \Rightarrow dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi$$
$$y = r\sin\varphi \Rightarrow dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$$

$$dx\wedge dy = (\cos\varphi\, dr - r\sin\varphi\, d\varphi)\wedge(\sin\varphi\, dr + r\cos\varphi\, d\varphi)$$
$$= (r\cos^2\varphi\, dr\wedge d\varphi - r^2\sin\varphi\, \underbrace{d\varphi\wedge dr}_{-}) = r\,dr\wedge d\varphi$$



Legendre transforms

$f(x,y)$: $\rightarrow$ zero-form

$$df(x,y) = \underbrace{\frac{\partial f}{\partial x}}_{u} dx + \underbrace{\frac{\partial f}{\partial y}}_{v} dy$$

$$df = \underset{u(x,y)}{u}\, dx + \underset{v(x,y)}{v}\, dy$$: diff form (1-form) expressed in the basis (dx, dy)

$g(u,y) = f(x,y) - ux$: $\rightarrow$ zero-form

$dg(u,y) = df(x,y) - d(ux)$ $\rightarrow$ 1 form

u is $u(x,y)$, so,

$$= df(x,y) - x\,du(x,y) - u(x,y)\,dx$$

$$= u(x,y)\,dx + v(x,y)\,dy - x\,du(x,y) - u(x,y)\,dx$$

$$dg(u,y) = v(x,y)\,dy - x\,du(x,y)$$

but because we want g to be function of u, y only

and $dg(u,y) = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial y} dy$

we must have

$$-\frac{\partial g}{\partial y} = x \quad ; \quad \frac{\partial g}{\partial y} = v(x,y)$$

in order to eliminate x.

- let $x \rightarrow \dot{q}$, $y \rightarrow q$

  $f(x,y) \rightarrow L(q, \dot{q})$

  $u \rightarrow p$, $g(u,y) \rightarrow H(p,q)$

  $\rightarrow$ Legendre transf. from mechanics

  $H(p,q) = L(q,\dot{q}) - p\dot{q}$

- Let $x \rightarrow V$ $y \rightarrow S$

  $f(x,y) \rightarrow U(S,V)$

  $u \rightarrow -P$, $g(u,y) \rightarrow H(S,P)$

  $\rightarrow$ legendre transf. from thermodyn.

  $$\underset{\text{enthalpy}}{H}(S,P) = \underset{\text{internal energy}}{U}(S,V) + VP$$

Ej ↲ Imagine a 3D vector space L with a basis dx_1 dx_2 dx_3.

a) Compute the gen. form of all possible p-forms.

b) compute $\underline{a} \wedge \underline{b}$ in that basis.

a)

$\Lambda^0 L = \mathbb{R}$: a (scalar)

$\Lambda^1 L = L$: $a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ (1-form)

$\Lambda^2 L$: $a_{12}\, dx_1 \wedge dx_2 + a_{23}\, dx_2 \wedge dx_3 + a_{31}\, dx_3 \wedge dx_1$

$\Lambda^3 L$: $a_{123}\, dx_1 \wedge dx_2 \wedge dx_3$

there are no more possible $\Lambda^p L$.

b) $(a_1 dx_1 + a_2 dx_2 + a_3 dx_3) \wedge (b_1 dx_1 + b_2 dx_2 + b_3 dx_3)$

$$= a_1 b_2\, dx_1 \wedge dx_2 + a_1 b_3\, dx_1 \wedge dx_3$$
$$+ a_2 b_1\, dx_2 \wedge dx_1 + a_2 b_3\, dx_2 \wedge dx_3$$
$$+ a_3 b_1\, dx_3 \wedge dx_1 + a_3 b_2\, dx_3 \wedge dx_2$$

$$= (a_1 b_2 - a_2 b_1)\, dx_1 \wedge dx_2 + (a_2 b_3 - a_3 b_2)\, dx_2 \wedge dx_3 + (a_3 b_1 - a_1 b_3)\, dx_1 \wedge dx_3$$

$\vec{a} \times \vec{b}$!

Ej: Compute $d(dx \wedge dy) = \underbrace{d(dx)}_{\text{rules} \Rightarrow 0} \wedge dy - dx \wedge \underbrace{d(dy)}_{\text{rules} \Rightarrow 0}$.

Problema: demuestre que $d(df) = 0$.

Considere $f \in \Lambda^0 L$, $f(x,y,z)$

$$\Rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \in \Lambda^1 L$$

$$\Rightarrow d(df) = d\left(\frac{\partial f}{\partial x} dx\right) + d\left(\frac{\partial f}{\partial y} dy\right) + d\left(\frac{\partial f}{\partial z}\right) dz$$

$$= \frac{\partial^2 f}{\partial x \partial x} \cancelto{0}{dx \wedge dx} + \frac{\partial^2 f}{\partial y \partial x} dy \wedge dx + \frac{\partial^2 f}{\partial z \partial x} dz \wedge dx$$

$$+ \frac{\partial^2 f}{\partial x \partial y} dx \wedge dy + \frac{\partial^2 f}{\partial y \partial y} \cancelto{0}{dy \wedge dy} + \frac{\partial^2 f}{\partial z \partial y} dz \wedge dy$$

$$+ \frac{\partial^2 f}{\partial x \partial z} dx \wedge dz + \frac{\partial^2 f}{\partial y \partial z} dy \wedge dz + \frac{\partial^2 f}{\partial z \partial z} \cancelto{0}{dz \wedge dz}$$

Si el orden de las derivadas es intercambiable, esto da cero.

Problema: demuestre que los coef. de $d(a_1 dx_1 + a_2 dx_2 + a_3 dx_3)$ dan $\vec{\nabla} \times \vec{a}$.

$$d(a_1 dx_1 + a_2 dx_2 + a_3 dx_3) = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2 \wedge dx_3$$

$$+ \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) dx_3 \wedge dx_1$$

$$+ \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

Problema: demuestre que d(2-forma) da los componentes de la divergencia.

$$d(a_{23} dx_2 \wedge dx_3 + a_{31} dx_3 \wedge dx_1 + a_{12} dx_1 \wedge dx_2)$$

$$= \left(\frac{\partial a_{23}}{\partial x_1} + \frac{\partial a_{31}}{\partial x_2} + \frac{\partial a_{12}}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$