Problem: show the normalization of the Legendre polynomials.

Solution:

We have to show that

$$\int_{-1}^{1} [P_\ell(x)]^2 \, dx = \frac{2}{2\ell+1}$$

We start from the recursion formula

$$\ell P_\ell(x) = x P_\ell'(x) - P_{\ell-1}'(x)$$

and substitute

$$\ell \int_{-1}^{1} [P_\ell(x)]^2 \, dx = \underbrace{\int_{-1}^{1} x P_\ell(x) P_\ell'(x) \, dx}_{\text{parts}} - \underbrace{\int_{-1}^{1} P_\ell(x) \underbrace{P_{\ell-1}'(x)}_{\substack{\text{this is a polynomial} \\ \text{of order } \ell-1 \\ \Rightarrow \text{orthogonal with } P_\ell(x)}} dx}_{=0}$$

$$\ell \underbrace{\int_{-1}^{1} [P_\ell(x)]^2}_{:= I} = \underbrace{\frac{x}{2}[P_\ell(x)]^2 \Big|_{-1}^{1}}_{=1 \text{ at boundaries}} - \frac{1}{2} \underbrace{\int_{-1}^{1} [P_\ell(x)]^2 \, dx}_{:= I}$$

$$\ell I = 1 - \frac{1}{2} I$$

$$\Rightarrow I = \frac{2}{2\ell+1} \quad \checkmark$$

Problem. Expand the function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

in a Legendre series, keeping only the first terms.

Solution:

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

Following the PDF on Sturm-Liouville, we have

$$c_0 = \frac{\langle f(x), P_0(x) \rangle}{\langle P_0(x), P_0(x) \rangle} = \frac{\int_0^1 dx}{\frac{2}{2\cdot 0+1}} = \frac{1}{2}$$

$$c_1 = \frac{\langle f(x), P_1(x) \rangle}{\langle P_1(x), P_1(x) \rangle} = \frac{\int_0^1 x \, dx}{\frac{2}{2\cdot 1+1}} = \frac{3}{4}$$

etc.

$$\Rightarrow f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + \cdots$$

Problem: obtain the normalization of the Bessel functions, using the change of variable $x = a r \Rightarrow \frac{d}{dx} = \frac{dr}{dx}\frac{d}{dr} = \frac{1}{a}\frac{d}{dr}$ where a are zeros of the Bessel function, i.e., $J_p(a) = 0$

Solution:

with the proposed transformation, we have

$$\underbrace{\frac{d}{dr}\left(r\frac{d}{dr}\right)y}_{p_{SL}(r)=r} - \underbrace{\frac{p^2}{r}y}_{s(r)=-\frac{p^2}{r}} + \underbrace{a^2}_{\lambda}\underbrace{r}_{w(r)=r}\, y = 0$$

in this case, the interval is $0 \leq x \leq 1$
$\longrightarrow 0 \leq r \leq 1$
because $J_p(a) = 0$.

From the PDF "Sturm Liouville" & Orthogonality... with weight function,
we substitute $p_{SL}(r) = r$, $a_{SL} \to 0$, $b_{SL} \to 1$, $\varphi_m(x) \to J_p(a r)$, $\varphi_n(x) \to J_p(b r)$, $w(r) = r$

careful! chain rule!

$$r\left[J_p(a r)\frac{d}{dr}J_p(b r) - \frac{d}{dr}J_p(a r)J_p(b r)\right]\Big|_0^1 = (a^2 - b^2)\int_0^1 dr\, r\, J_p(a r) J_p(b r)$$

when we evaluate in $r=0$, the r cancels everything, so the surviving elements are

chain rule / chain rule

$$\frac{J_p(a)\, b J_p'(b) - a J_p'(a) J_p(b)}{a^2 - b^2} = \int_0^1 dr\, r\, J_p(a r) J_p(b r)$$

if we take the limit $b \to a$, we get $0/0$.
We apply L'Hôpital's rule:

$$\int_0^1 dr\, r\, J_p(a r) J_p(b r) = \lim_{b\to a} \frac{\frac{\partial}{\partial b}\left[J_p(a)\, b J_p'(b) - a J_p'(a) J_p(b)\right]}{\frac{\partial}{\partial b}(a^2 - b^2)}$$

$$= \lim_{b\to a} \frac{\underbrace{\cancel{J_p(a)}}_{\text{boundary condition}}^{0}\frac{d}{db}[b J_p'(b)] - a J_p'(a)\, J_p'(b)}{-2b}$$

$$= -a\left[J_p'(a)\right]^2 / (-2a) = \frac{1}{2}\left[J_p'(a)\right]^2$$

Now we use the recursion formula

$$\frac{d}{dx}\left(\frac{J_p(x)}{x^p}\right) = -\frac{J_{p+1}(x)}{x^p}$$

$$x = Kr \Rightarrow \frac{d}{dx} = \frac{1}{K}\frac{d}{dr}$$

$$\Rightarrow \frac{1}{K}\frac{d}{dr}\left(\frac{J_p(Kr)}{(Kr)^p}\right) = -\frac{J_{p+1}(Kr)}{(Kr)^p}$$

$$\frac{1}{K}\left[\frac{K J_p'(Kr)}{(Kr)^p} - \frac{J_p(Kr)}{(Kr)^{2p}}\cdot K\right] = -\frac{J_{p+1}(Kr)}{(Kr)^p}$$

at $r=1$, and applying bound. cond.,

$$\cancel{\frac{1}{K}}\frac{\cancel{K} J_p'(K)}{\cancel{K^p}} = -\frac{J_{p+1}(K)}{\cancel{K^p}}$$

$$\Rightarrow J_p'(\kappa) = -J_{p+1}(\kappa)$$

so,

$$\int_0^1 dr\, r\, J_p(a\, r) J_p(b\, r) = \frac{1}{2}[J_{p+1}(a)]^2$$

Problem. Prove the recurrence relations

$$\frac{d}{dx}(x^{-n} Z_n) = -x^{-n} Z_{n+1} \quad ; \quad \frac{d}{dx}(x^n Z_n) = x^n Z_{n-1}$$

where Z_n are solutions of the Bessel equation.

Solution: This can be done in many ways. This is one:

Write the ODE in a factored form

$$\left(\frac{d}{dx} + \frac{n+1}{x}\right)\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n = -Z_n$$

Let's define

$$\bar{Z}_n = \left(\frac{d}{dx} - \frac{n}{x}\right) Z_n$$

and if we apply the operator $\left(\frac{d}{dx} - \frac{n}{x}\right)$ on the factored ODE we form the expression

$$\left(\frac{d}{dx} - \frac{n}{x}\right)\left(\frac{d}{dx} + \frac{n+1}{x}\right)\underbrace{\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n}_{\bar{Z}_n} = -\underbrace{\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n}_{\bar{Z}_n}$$

$$\Rightarrow \left(\frac{d}{dx} - \frac{n}{x}\right)\left(\frac{d}{dx} + \frac{n+1}{x}\right)\bar{Z}_n = -\bar{Z}_n$$

$$\Rightarrow \frac{d^2}{dx^2}\bar{Z}_n + \frac{1}{x}\frac{d}{dx}\bar{Z}_n + \left[1 - \frac{(n+1)^2}{x^2}\right]\bar{Z}_n = 0$$

This is the Bessel ODE of order $n+1$

$$\Rightarrow \left(\frac{d}{dx} - \frac{n}{x}\right) Z_n = \bar{Z}_n = -Z_{n+1}$$

$$\Rightarrow \left[\frac{dZ_n}{dx} - \frac{n}{x} Z_n = -Z_{n+1}\right] \cdot x^{-n}$$

$$\Rightarrow \frac{d}{dx}(x^{-n} Z_n) = -x^{-n} Z_{n+1}$$

For the second identity, we try another way, using the explicit series of $J_p(x)$

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\frac{x^{2n+2p}}{2^{2n+p}}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+2p)}{\Gamma(n+1)\Gamma(n+p+1)}\frac{x^{2n+2p-1}}{2^{2n+p}}$$

but $\Gamma(n+p+1) = (n+p)\Gamma(n+p)$, so we use this to cancel the $(n+p)$ in the numerator:

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)}\frac{x^{2n+2p-1}}{2^{2n+p-1}}$$

the 2 in the numerator

dividing both sides by x^p

$$\frac{1}{x^p}\frac{d}{dx}\left[x^p J_p(x)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)}\left(\frac{x}{2}\right)^{2n+p-1} = J_{p-1}(x)$$

Problem. In a diffraction problem for a circular apperture, the integral

$$\Phi = \int_0^a \int_0^{2\pi} e^{ibr\cos\theta}\, d\theta\; r\, dr$$

appears, where a, b are constants. Evaluate the integral using Bessel's functions.

The integral representation of the Bessel functions is

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta \quad \text{or} \quad J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\,d\theta$$

for $n=0$,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta)\,d\theta =$$

$\cos(x\sin\theta)$ repeats itself in all four quadrants

ver notas de Heidy, Ejemplos 7.1.2, 7.1.3

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta$$

because of Euler's formula and $\int_0^{2\pi}\sin(x\sin\theta)\,d\theta = 0$

then

$$\Phi = 2\pi\int_0^a J_0(br)\, r\, dr$$

using the recurrence formula $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ with $n=1$,

$$\Phi = \frac{2\pi a b}{b^2} J_1(ab).$$

Problem. Show that the Ansatz $\psi(z) = e^{-\frac{z^2}{2}} v(z)$ on the differential equation

$$\frac{d^2}{dz^2}\psi(z) + \left(\frac{2E}{\hbar\omega} - z^2\right)\psi(z) = 0 \qquad (1)$$

with $E, \hbar, \omega$ being constants, transforms it into the Hermite ODE.

Solution:

$$\psi(z) = e^{-z^2/2} v(z) \quad (2)$$

$$\frac{d\psi}{dz} = -z e^{-z^2/2} v(z) + e^{-\frac{z^2}{2}} \frac{dv(z)}{dz} = e^{-z^2/2}\left(-zv + \frac{dv}{dz}\right)$$

$$\frac{d^2\psi}{dz^2} = -ze^{-z^2/2}\left(-zv + \underline{\frac{dv}{dz}}\right) + e^{-z^2/2}\left(-v - z\underline{\frac{dv}{dz}} + \frac{d^2v}{dz^2}\right)$$

$$= e^{-z^2/2}\left(z^2 v - 2z\frac{dv}{dz} - v + \frac{d^2v}{dz^2}\right) \qquad (3)$$

Substituting (3), (2) in (1), we see that the factor $e^{-z^2/2}$ cancels out:

$$\cancel{z^2}v - 2z\frac{dv}{dz} - v + \frac{d^2v}{dz^2} + \frac{2E}{\hbar\omega}v - \cancel{z^2}v = 0$$

$$\Rightarrow v'' - 2zv' + \left(\frac{2E}{\hbar\omega} - 1\right)v = 0 \quad \checkmark$$

Problem: Show the recurrence formula

$$H_n'(x) = 2n H_{n-1}(x)$$

Solution:

We start from the generating function

$$\Phi(x,h) = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!} = e^{-h^2+2hx}$$

And try

$$\frac{\partial\Phi}{\partial x} = 2he^{-h^2+2hx} = 2h\Phi = 2h$$

we substitute Φ and $\frac{\partial\Phi}{\partial x}$ by the series.

$$\Phi(x,h) = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}; \quad \frac{\partial\Phi}{\partial x} = \sum_{n=0}^{\infty} H_n'(x)\frac{h^n}{n!}$$

$$\sum_{n=0}^{\infty} H_n'(x)\frac{h^n}{n!} = \sum_{n=0}^{\infty} 2h\, H_n(x)\frac{h^n}{n!}$$

$$\downarrow$$

$$2\sum_{n=0}^{\infty} H_n(x)\frac{h^{n+1}}{n!}$$

$$\Big\downarrow \text{ let } m = n+1 \Rightarrow \text{when } n=0,\ m=1$$

$$\sum_{n=0}^{\infty} H_n'(x)\frac{h^n}{n!} = 2\sum_{m=1}^{\infty} H_{m-1}(x)\frac{h^m}{(m-1)!}$$

$$\Big\downarrow m=n, \text{ recursivity of factorial}$$

$$\sum_{m=0}^{\infty} H_m'(x)\frac{h^m}{m(m-1)!} = 2\sum_{m=1}^{\infty} H_{m-1}(x)\frac{h^m}{(m-1)!}$$

Finally, for $m \geq 1$, $\frac{H_m'(x)}{m} = 2H_{m-1}(x) \Rightarrow H_n'(x) = 2n H_{n-1}(x)$ ✓

Problem: Write an integral representation of the Hermite polynomials.

Solution:

Start from the generating function

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

and multiply by t^{-m-1}:

$$t^{-m-1} e^{-t^2+2tx} = \sum_{n=0}^{\infty} t^{-m-1+n} \frac{H_n(x)}{n!}$$

integrate in the complex plane around the origin:

$$\oint t^{-m-1} e^{-t^2+2tx} dt = \oint \sum_{n=0}^{\infty} t^{-m-1+n} \frac{H_n(x)}{n!} dt$$

With poles at $t=0$, we find with the residue theorem that only the term $n=m$ survives:

$$\oint t^{-m-1} e^{-t^2+2tx} dt = 2\pi i \frac{H_m(x)}{m!}$$

$$\Rightarrow H_m(x) = \frac{m!}{2\pi i} \oint t^{-m-1} e^{-t^2+2tx} dt$$

Problem: find the normalization of the Hermite polynomials.

Solution:

We start this time from the generating function (we multiply two of them, one with the variable t and one with the variable s) and build the following expression

$$e^{-x^2} e^{-s^2+2sx} e^{-t^2+2tx} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-x^2} H_m(x) H_n(x) \frac{s^m t^n}{m!\, n!}$$

now we integrate over the interval $]-\infty,\infty[$ and from the ortogonality, we know the terms other than $m=n$ will be zero.

$$\sum_{n=0}^{\infty} \frac{(st)^n}{n!\, n!} \int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = \int_{-\infty}^{\infty} e^{-x^2-s^2+2sx-t^2+2tx-2st+2st} dx$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-(x-s-t)^2}}_{\text{Known integral}} e^{2st} dx = \sqrt{\pi}\, e^{2st} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n (st)^n}{n!}$$

Examing coefficients term by term of the power st we obtain

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = 2^n \sqrt{\pi}\, n!$$