Problem: show the normalization of the Legendre polynomials.

Solution method I: using the recursion formula.

We have to show that

$$\int_{-1}^{1} [P_\ell(x)]^2 \, dx = \frac{2}{2\ell+1}$$

We start from the recursion formula

$$\ell P_\ell(x) = x P_\ell'(x) - P_{\ell-1}'(x)$$

and substitute

$$\ell \int_{-1}^{1} [P_\ell(x)]^2 \, dx = \int_{-1}^{1} x P_\ell(x) P_\ell'(x) \, dx - \underbrace{\int_{-1}^{1} P_\ell(x) \underbrace{P_{\ell-1}'(x)}_{\substack{\text{this is a polynomial} \\ \text{of order } \ell-1 \\ \Rightarrow \text{orthogonal with } P_\ell(x)}} dx}_{=0}$$

parts

$$\ell \underbrace{\int_{-1}^{1} [P_\ell(x)]^2}_{:= I} = \underbrace{\frac{x}{2}[P_\ell(x)]^2 \Big|_{-1}^{1}}_{=1 \text{ at boundaries}} - \frac{1}{2}\underbrace{\int_{-1}^{1} [P_\ell(x)]^2 \, dx}_{:= I}$$

$$\ell I = 1 - \frac{1}{2} I$$

$$\Rightarrow I = \frac{2}{2\ell+1} \quad ✓$$

Solution method II: using the generating function.

The generating function is $\displaystyle \frac{1}{\sqrt{1-2t\mu+t^2}} = \sum_{\ell=0}^{\infty} P_\ell(\mu)\, t^\ell$

We now square it and integrate over μ on both sides in order to form the normalization integral inside of the sum

$$\underbrace{\int_{-1}^{1} \frac{1}{1-2t\mu+t^2} \, d\mu}_{= \frac{1}{t} \ln \frac{1+t}{1-t}} = \sum_{\ell=0}^{\infty} t^{2\ell} \int_{-1}^{1} [P_\ell(\mu)]^2 \, d\mu$$

Now we expand the left hand side in order to also form a series there. Our goal is to compare coefficients of the expansion in powers of t in the end., GANDREOLIVA.org

$$\frac{1}{t}\ln\frac{1+t}{1-t} = 2+\frac{2}{3}t^2+\frac{2}{5}t^4+\cdots+\frac{2}{2\ell+1}t^{2\ell}+\cdots$$
$$=\sum_{\ell=0}^{\infty}t^{2\ell}\cdot\frac{2}{2\ell+1}$$

Tables:

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$$
$$=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

$$\sum_{\ell=0}^{\infty}t^{2\ell}\cdot\frac{2}{2\ell+1}=\sum_{\ell=0}^{\infty}t^{2\ell}\int_{-1}^{1}[P_\ell(\mu)]^2\,d\mu$$

$$\Rightarrow \int_{-1}^{1}[P_\ell(\mu)]^2\,d\mu=\frac{2}{2\ell+1} \qquad ▨$$

Problem: compute the expansion of the function

$$f(x)=\sqrt{\frac{1-x}{2}}$$

as a series in terms of the Legendre polynomials.

Solution: the general procedure for computing the coefficients of the series expansion (inner product from the left –> orthogonality –> solving for the coefficients) can also be applied to the generating function. We try this calculation way for this exercise:

Multiplying the generating function by $\sqrt{\frac{1-x}{2}}$ and integrating,

$$\underbrace{\int_{-1}^{1}\frac{\sqrt{1-x}}{\sqrt{2}}\frac{1}{\sqrt{1-2xt+t^2}}\,dx}_{\text{rationalize \& substitute}} = \sum_{n=0}^{\infty}t^n\int_{-1}^{1}\sqrt{\frac{1-x}{2}}P_n(x)\,dx$$

to obtain

$$=\frac{1}{2t}\left[1+t-\frac{(1-t)^2}{2\sqrt{t}}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}\right]$$

expanding the l.h.s. as a power series of t,

$$\cdots\Rightarrow \frac{4}{3}-4\sum_{n=1}^{\infty}\frac{t^n}{(4n^2-1)(2n+3)}=\sum_{n=0}^{\infty}t^n\int_{-1}^{1}\sqrt{\frac{1-x}{2}}P_n(x)\,dx$$

comparing coefficients,

$$\int_{-1}^{1}\sqrt{\frac{1-x}{2}}P_0(x)\overset{!}{=}\frac{4}{3}$$
$$\int_{-1}^{1}\sqrt{\frac{1-x}{2}}P_n(x)\overset{!}{=}-\frac{4}{(4n^2-1)(2n+3)}$$

$$\Rightarrow \sqrt{\frac{1-x}{2}}=\frac{2}{3}P_0(x)-2\sum_{n=1}^{\infty}\frac{P_n(x)}{(2n-1)(2n+3)} \qquad ▨$$

Problem: compute the integral

$$\int_0^1 P_\ell(\mu)\, d\mu$$

Solution: we use the generating function and a subsequent expansion in powers of t in order to get the integral

$$\int_0^1 \frac{1}{\sqrt{1-2t\mu+t^2}}\, d\mu = \int_0^1 \sum_{\ell=0}^{\infty} t^\ell P_\ell(\mu)\, d\mu$$

Tables:

Binomial series:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

l. h. s.:

$$\int_0^1 \frac{d\mu}{\sqrt{1-2t\mu+t^2}} = -\frac{1}{t}\sqrt{1-2t\mu+t^2}\Big|_0^1 = \underbrace{\frac{1}{t}\left[\sqrt{1+t^2}-(1-t)\right]}_{\text{expansion as a power series}}$$

$$= 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} t^{2n-1}$$

$$\Rightarrow \quad 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} t^{2n-1} = \sum_{\ell=0}^{\infty} t^\ell \int_0^1 P_\ell(\mu)\, d\mu$$

$$\Rightarrow \quad 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} t^{2n-1} = 1 + \sum_{\ell=1}^{\infty} t^\ell \int_0^1 P_\ell(\mu)\, d\mu$$

comparing,

$$\int_0^1 P_\ell(\mu)\, d\mu = \begin{cases} 1, & \ell = 0 \\ 0, & \ell \text{ even (but not 0)} \\ \binom{1/2}{n}, & \ell = 2n-1 \quad (\ell \text{ odd}) \end{cases}$$

$\blacksquare$

Problem: Obtain a series expansion of the function

$$f(x) = H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$$

in terms of the Legendre polynomials.

Solution:

The coefficients are given by

$$c_\ell = \frac{\langle f(x), P_\ell(x)\rangle}{N_\ell^2} = \frac{2\ell+1}{2}\int_{-1}^{1} H(x) P_\ell(x)\, dx = \frac{2\ell+1}{2}\int_0^1 P_\ell(x)\, dx$$

which is the previous exercise, and can be re-written in several ways.

To test the expansion, we plot the first 100 coefficients, that is

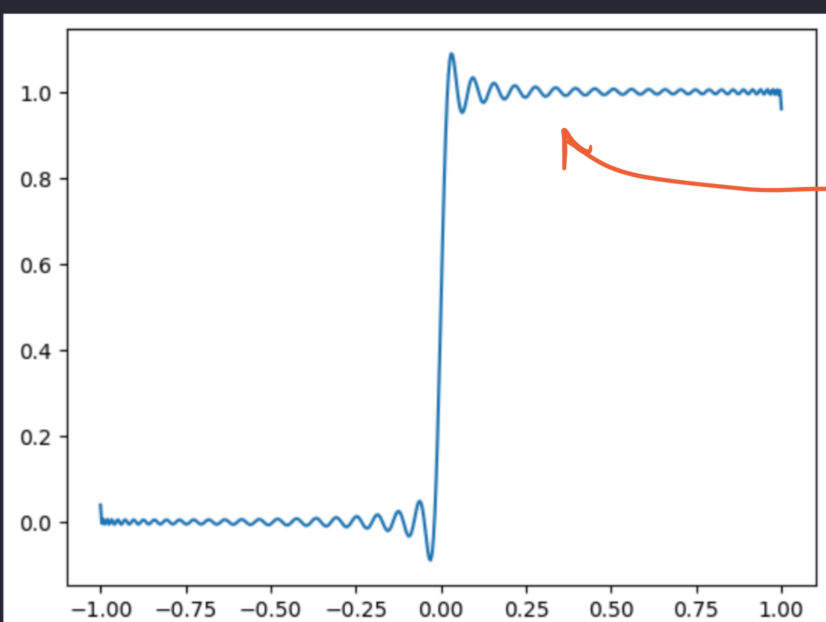
$$H_\ell(x) \approx \sum_{\ell=0}^{100} c_\ell P_\ell(x)$$

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import gamma, legendre
```
✓ 0.0s

```python
def coeff(l):
    norm = (2*l + 1)/2
    if l == 0: return 1 * norm
    elif l % 2 == 0: return 0
    else:
        n = (l + 1)//2
        return norm * gamma(1/2+1)/(gamma(n+1)*gamma(1/2-n+1))
```
✓ 0.0s

```python
x = np.linspace(-1,1,500)
order = 100
series_approx = 0
for i in range(order):
    series_approx += coeff(i)*legendre(i)(x)
plt.plot(x, series_approx)
```
✓ 0.0s

```
[<matplotlib.lines.Line2D at 0x11ab52600>]
```



artifact
very slow convergence!

Problem: show that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for} \quad m<n.$$

Solution:

Using the Rodrigues' formula,

$$\int_{-1}^{1} x^m \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\, dx$$

parts once:

$$= \cancel{\frac{x^m}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\Big|_{-1}^{1}} - \int_{-1}^{1}\frac{d}{dx}(x^m)\frac{1}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx$$

0 boundary conditions of Legendre

⋮

parts n times

$$= (-1)^n\int_{-1}^{1}\frac{d^n}{dx^n}(x^m)\frac{1}{2^n n!}(x^2-1)^n\, dx$$

if $n>m$, this term will be zero. ☑

Problem: Expand the function

$$f(x) = \begin{cases} a, & x<0 \\ 0, & x\geq 0 \end{cases}$$

as a Legendre series in the interval [-1,1]

Solution: We compute the coefficients in a naïve way first, just as we did in a previous exercise. We want the integral $\int_{-1}^{1} f(x) P_\ell(x) dx$.

$$\int_{-1}^{0}\frac{1}{\sqrt{1-2t\mu+t^2}}\,d\mu = \int_{-1}^{0}\sum_{\ell=0}^{\infty} t^\ell P_\ell(\mu)\, d\mu$$

Tables:

Binomial series:

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$$

l. h. s.:

$$\int_{-1}^{0}\frac{d\mu}{\sqrt{1-2t\mu+t^2}} = -\frac{1}{t}\sqrt{1-2t\mu+t^2}\Big|_{-1}^{0} = -\frac{1}{t}\left[\sqrt{1+t^2} - \sqrt{1+2t+t^2}\right]$$

$$= \frac{1}{t}\left[1+t-\sqrt{1+t^2}\right]$$

$$= \frac{1}{t}\left[1+t-\sum_{k=0}^{\infty}\binom{1/2}{k}t^{2k}\right]$$

↳ first term:

$$\binom{1/2}{0}t^0 = \frac{\Gamma(1/2+1)}{\Gamma(0+1)\Gamma(1/2-0+1)} = \frac{\Gamma(3/2)}{1\cdot\Gamma(3/2)} = 1$$

$$= \frac{1}{t}\left[\cancel{1}+t-\cancel{1}-\sum_{k=1}^{\infty}\binom{1/2}{k}t^{2k}\right]$$

$$= 1-\sum_{k=1}^{\infty}\binom{1/2}{k}t^{2k-1}$$

$$\Rightarrow \quad 1 - \sum_{k=1}^{\infty} \binom{1/2}{k} t^{2k-1} = \sum_{\ell=0}^{\infty} t^{\ell} \underbrace{\int_{-1}^{0} P_\ell(\mu)\, d\mu}_{:= I_\ell}$$

$$I_\ell = \begin{cases} 1, & \ell = 0 \\ 0, & \ell = 2k \quad (\text{even}) \\ -\binom{1/2}{k}, & \ell = 2k-1 \quad (\text{odd}) \end{cases} \qquad \text{with } k \in \mathbb{N}.$$

Therefore, the expansion of the function $f(x) = \begin{cases} a, & x < 0 \\ 0, & x \geq 0 \end{cases}$

is given by $f(x) = a \sum_{\ell=0}^{\infty} I_\ell \cdot \frac{2\ell+1}{2} \cdot P_\ell(x)$

With the expansion we had done before, we could also have predicted this result by simply writing

$f(x) = a \cdot [1 - H(x)]$ ▨

Associated Legendre equation

The associated Legendre equation is a generalization of the Legendre equation given by

$$\frac{d}{dx}\left((1-x^2)\,y'(x)\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right) y(x) = 0 \quad , \quad m^2 \leq \ell^2$$

for the case $m=0$ we recover the Legendre ODE.

We propose the Ansatz

$$y(x) = (1-x^2)^{m/2}\, v(x)$$

and after substitution, one can show that this yields

$$(1-x^2)\,v''(x) - 2(m+1)\,x\,v'(x) + [\ell(\ell+1) - m(m+1)]\,v(x) = 0 \qquad (*)$$

If we then differentiate both sides of this equation, we get

$$(1-x^2)\,[v'(x)]'' - 2(m+2)\,x\,[v'(x)]' + (\ell(\ell+1) - (m+1)(m+2))\,[v'(x)] = 0$$

We notice that this last equation is the same as equation $(*)$ with the substitutions $m \longrightarrow m+1$, $v(x) \longrightarrow v'(x)$.

Now we notice that we have

- $y(x) = v(x) = P_\ell(x)$ is a solution for $m=0$ [eq. $(*)$ with $m=0$ is the Legendre ODE]
- $v(x) = P_\ell'(x)$ " " " " $m=1$
  $\Rightarrow y(x) = (1-x^2)^{1/2} P_\ell'(x)$
- $v(x) = P_\ell''(x)$ " " " " $m=2$
  $\Rightarrow y(x) = (1-x^2)\, P_\ell''(x)$

$\vdots$

Then we define:

The **Associated Legendre Functions** as

$$y(x) := P_\ell^m(x) := \overset{\text{conventional}}{(-1)^m}\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m} P_\ell(x) \quad , \quad 0 \leq m \leq \ell$$

and the **Associated Legendre Polynomials** as

$$v(x) := \mathcal{P}_\ell^m(x) := (-1)^m \frac{d^m}{dx^m} P_\ell(x)$$

WARNING: the notation and normalization is different in every book!

Problem: find the Rodrigue's formula for the assoc. Legendre functions.

$$P_\ell^m = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m}\left(\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right)$$

$$= \frac{(-1)^m (1-x^2)^{m/2}}{2^\ell \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \quad , \quad \ell \geq 0,\ |m| \leq \ell,\ m \in \mathbb{Z},\ \ell \in \mathbb{N}_0$$

Problem. Obtain a recurrence relation of the associated Legendre polynomials for the index m.

Solution: starting with the ODE

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]y = 0$$

we introduce the Ansatz $y = (x^2-1)^{m/2} v$ and obtain

$$(1-x^2)v'' - 2(m+1)xv' + (\ell-m)(\ell+m+1)v = 0$$

We now set $v = d^m P_\ell(x)/dx^m$
and multiply by $(x^2-1)^{m/2}$

$$\left[(1-x^2)\underbrace{\frac{d^{m+2}}{dx^{m+2}}P_\ell(x)}_{P_\ell^{m+2}(x)} - 2(m+1)x\frac{d^{m+1}}{dx^{m+1}}P_\ell(x) + (\ell-m)(\ell+m+1)P_\ell(x) = 0\right]\cdot(x^2-1)^{m/2}$$

$$\Rightarrow P_\ell^{m+2}(x) + \frac{2(m+1)x}{(x^2-1)^{1/2}}P_\ell^{m+1}(x) - (\ell-m)(\ell+m+1)P_\ell^m(x) = 0$$

Problem. Compute the normalization of the associated Legendre functions.

Hints: a) A recursion relation satisfied by the assoc. Legendre funct. is

$$(2\ell+1)\mu P_\ell^m = (\ell-m+1)P_{\ell+1}^m + (\ell+m)P_{\ell-1}^m \quad ; \quad P_\ell^m = P_\ell^m(\mu)$$

b) It can be shown that

$$P_m^m(\mu) = \frac{(2m)}{m!\,2^m}\sin^m\theta$$

c) The following integral is known:

$$\int_0^\pi \sin^{2m+1}\theta\, d\theta = \frac{2^{2m+1}(m!)^2}{(2m+1)!}$$

Solution: We want to compute the integral

$$\int_{-1}^1 [P_\ell^m]^2 d\mu \quad , \text{ where } \mu = \cos\theta \; , \; P_\ell^m = P_\ell^m(\mu)$$

We start from the recurrence relation to form this integral.

$$\int_{-1}^1\left[\left[(2\ell+1)\mu P_\ell^m = (\ell-m+1)P_{\ell+1}^m + (\ell+m)P_{\ell-1}^m\right]\cdot P_{\ell-1}^m\right]d\mu$$

using orthogonality, $\int_{-1}^1 P_\ell^m P_{\ell'}^m d\mu = 0$,

$$\Rightarrow (2\ell+1)\int_{-1}^1 \mu P_\ell^m \underbrace{P_{\ell-1}^m}_{} d\mu = (\ell+m)\int_{-1}^1 \left[P_{\ell-1}^m\right]^2 d\mu$$

use recurrence formula again to substitute

$$\Rightarrow (\ell-m)\frac{(2\ell+1)}{(2\ell-1)}\int_{-1}^1 \left[P_\ell^m(\mu)\right]^2 d\mu = (\ell+m)\int_{-1}^1 \left[P_{\ell-1}^m(\mu)\right]^2 d\mu$$

Repeating this procedure until $\ell = m$...

$$\cdots \longrightarrow \int_{-1}^{1} [P_\ell^m(\mu)]^2 d\mu = \frac{2m+1}{2\ell+1} \frac{(\ell+m)!}{(2m)!(\ell-m)!} \int_{-1}^{1} [P_m^m(\mu)]^2 d\mu$$

Using the other two hints, we obtain

$$\int_{-1}^{1} [P_\ell^m(\mu)]^2 d\mu = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$$

Bessel's equation

When we solve the ODE

$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

by the series solution method, we find an explicit expression for the Bessel functions of the 1st kind of order p:

$$y(x) \equiv J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

for integer p > 0. For integer p < 0, the $\Gamma(n-p+1)$ function diverges if $n \leq p-1$
Then, the first terms of the series are going to be zero $(1/\infty \to 0)$
until the series starts as in the positive case. One can show that

$$J_{-p}(x) = (-1)^p J_p(x)$$

so that the solution with positive and negative integer p are not linearly independent. We need to find another solution

Bessel functions of the second kind
For integer p, the function

$$Y_p(x) = \lim_{\nu \to p} \frac{\cos(\pi\nu) J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)} \qquad (\text{also} := N_p(x))$$

provides a linearly independent solution to the Bessel ODE. (Neumann or Weber function)
In summary, the full solution is

$$y(x) = A J_p(x) + B Y_p(x), \quad p \in \mathbb{Z}$$

Note that if p is not an integer, we can use the negative p as a linearly independent solution and in that case, the general solution is

$$y(x) = A J_p(x) + B J_{-p}(x), \quad p \text{ non-integer}$$

Bessel ODE as a Sturm-Liouville problem
The differential operator of the Bessel ODE can be written as

$$x^2 y'' + x y' + (x^2 - p^2) y = 0 \quad \Rightarrow \quad x \frac{d}{dx}\left(x \frac{d}{dx}\right) y + (x^2 - p^2) y = 0$$

dividing by x,

$$\frac{d}{dx}\left(x \frac{d}{dx}\right) y - \frac{p^2}{x} y + 1 \cdot x \cdot y = 0$$

now we can identify

$p_{SL}(x) = x$ (Sturm-Liouville) $\quad s(x) = p^2/x$ (order of the function) $\quad \lambda = 1 \quad w(x) = x$

Problem: obtain the normalization of the Bessel functions, using the change of variable $x = a r \Rightarrow \frac{d}{dx} = \frac{dr}{dx}\frac{d}{dr} = \frac{1}{a}\frac{d}{dr}$
where a are zeros of the Bessel function, i.e., $J_p(a) = 0$

Solution:

in this case, the interval is $0 \leq x \leq 1$
$\longrightarrow 0 \leq r \leq 1$
because $J_p(a) = 0$.

with the proposed transformation, we have

$$\frac{d}{dr}\left(r\frac{d}{dr}\right)y - \frac{p^2}{r}y + a^2 r y = 0$$

$p_{SL}(r) = r$, $s(r) = -\frac{p^2}{r}$, λ, $w(r) = r$

From the PDF "Sturm Liouville" & Orthogonality... with weight function,
we substitute $p_{SL}(r) = r$, $a_{SL} \to 0$, $b_{SL} \to 1$, $\varphi_m(x) \to J_p(ar)$, $\varphi_n(x) \to J_p(br)$, $w(r) = r$

careful! chain rule!

$$r\left[J_p(ar)\frac{d}{dr}J_p(br) - \frac{d}{dr}J_p(ar)J_p(br)\right]\Big|_0^1 = (a^2 - b^2)\int_0^1 dr\, r\, J_p(ar)J_p(br)$$

when we evaluate in $r=0$, the r cancels everything, so the surviving elements are

chain rule, chain rule

$$\frac{J_p(a)\, b\, J_p'(b) - a\, J_p'(a)\, J_p(b)}{a^2 - b^2} = \int_0^1 dr\, r\, J_p(ar)J_p(br)$$

if we take the limit $b \to a$, we get $0/0$.
We apply L'Hôpital's rule:

$$\int_0^1 dr\, r\, J_p(ar)J_p(br) = \lim_{b\to a}\frac{\frac{\partial}{\partial b}\left[J_p(a)\, b\, J_p'(b) - a\, J_p'(a)\, J_p(b)\right]}{\frac{\partial}{\partial b}\left(a^2 - b^2\right)}$$

$$= \lim_{b\to a}\frac{\cancel{J_p(a)}^{\,0}\frac{d}{db}\left[bJ_p(b)\right] - a\, J_p'(a)\, J_p'(b)}{-2b}$$

boundary condition

$$= -a\left[J_p'(a)\right]^2 / (-2a) = \frac{1}{2}\left[J_p'(a)\right]^2$$

Now we use the recursion formula

$$\frac{d}{dx}\left(\frac{J_p(x)}{x^p}\right) = -\frac{J_{p+1}(x)}{x^p}$$

$$x = Kr \Rightarrow \frac{d}{dx} = \frac{1}{K}\frac{d}{dr}$$

$$\Rightarrow \frac{1}{K}\frac{d}{dr}\left(\frac{J_p(Kr)}{(Kr)^p}\right) = -\frac{J_{p+1}(Kr)}{(Kr)^p}$$

$$\frac{1}{K}\left[\frac{K J_p'(Kr)}{(Kr)^p} - \frac{J_p(Kr)}{(Kr)^{2p}}\cdot K\right] = -\frac{J_{p+1}(Kr)}{(Kr)^p}$$

at $r=1$, and applying bound. cond.,

$$\frac{1}{\cancel{K}}\frac{\cancel{K} J_p'(K)}{\cancel{K^p}} = -\frac{J_{p+1}(K)}{\cancel{K^p}}$$

$$\Rightarrow J_p'(K) = -J_{p+1}(K)$$

so,

$$\int_0^1 dr\, r\, J_p(a\, r) J_p(b\, r) = \frac{1}{2}\left[J_{p+1}(a)\right]^2$$

Generalized Bessel ODE

Bessel functions form a family of solutions of the general ODE

$$y'' + \frac{1-2a}{x} y' + \left[(b c x^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2}\right] y = 0$$

with constant a, b, c, p

The solution of this equation is $y = x^a Z_p(b x^c)$

Z_p is J, N or a combination of them.

Problem: prove the recurrence relations

$$\frac{d}{dx}(x^{-n} Z_n) = -x^{-n} Z_{n+1} \quad ; \quad \frac{d}{dx}(x^n Z_n) = x^n Z_{n-1}$$

where Z_n is a solution of the Bessel equation.

This can be done in many ways. This is one:

Write the ODE in a factored form

$$\left(\frac{d}{dx} + \frac{n+1}{x}\right)\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n = -Z_n$$

Let's define

$$\bar{Z}_n = \left(\frac{d}{dx} - \frac{n}{x}\right) Z_n$$

and if we apply the operator $\left(\frac{d}{dx} - \frac{n}{x}\right)$ on the factored ODE we form the expression

$$\left(\frac{d}{dx} - \frac{n}{x}\right)\left(\frac{d}{dx} + \frac{n+1}{x}\right)\underbrace{\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n}_{\bar{Z}_n} = -\underbrace{\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n}_{\bar{Z}_n}$$

$$\Rightarrow \left(\frac{d}{dx} - \frac{n}{x}\right)\left(\frac{d}{dx} + \frac{n+1}{x}\right) \overline{Z}_n = -\overline{Z}_n$$

$$\Rightarrow \frac{d^2}{dx^2}\overline{Z}_n + \frac{1}{x}\frac{d}{dx}\overline{Z}_n + \left[1 - \frac{(n+1)^2}{x^2}\right]\overline{Z}_n = 0$$

This is the Bessel ODE of order $n+1$

$$\Rightarrow \left(\frac{d}{dx} - \frac{n}{x}\right) Z_n = \overline{Z}_n = -Z_{n+1}$$

$$\Rightarrow \left[\frac{dZ_n}{dx} - \frac{n}{x} Z_n = -Z_{n+1}\right] \cdot x^{-n}$$

$$\Rightarrow \frac{d}{dx}\left(x^{-n} Z_n\right) = -x^{-n} Z_{n+1}$$

For the second identity, we try another way, using the explicit series of $J_p(x)$

$$\frac{d}{dx}\left[x^p J_p(x)\right] = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n+2p}}{2^{2n+p}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p)}{\Gamma(n+1)\Gamma(n+p+1)} \frac{x^{2n+2p-1}}{2^{2n+p}}$$

but $\Gamma(n+p+1) = (n+p)\Gamma(n+p)$, so we use this to cancel the $(n+p)$ in the numerator:

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \frac{x^{2n+2p-1}}{2^{2n+p-1}}$$

the 2 in the numerator

dividing both sides by x^p

$$\frac{1}{x^p}\frac{d}{dx}\left[x^p J_p(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p)} \left(\frac{x}{2}\right)^{2n+p-1} = J_{p-1}(x)$$

Problem: verify that the Bessel ODE

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

can be factored as

$$\left(\frac{d}{dx} + \frac{n+1}{x}\right)\left(\frac{d}{dx} - \frac{n}{x}\right) Z_n = -Z_n$$

Solution:

We apply the first operator,

$$\left(\frac{d}{dx} + \frac{n+1}{x}\right)\left[\frac{dZ_n}{dx} - \frac{nZ_n}{x}\right] = -Z_n$$

We apply the second operator,

$$\frac{d^2Z_n}{dx^2} - \frac{d}{dx}\left[\frac{nZ_n}{x}\right] + \frac{(n+1)}{x}\frac{dZ_n}{dx} - \frac{(n+1)n}{x^2}Z_n = -Z_n$$

$$\frac{d^2Z_n}{dx^2} - \frac{n}{x}\frac{dZ_n}{dx} + \frac{nZ_n}{x^2} + \frac{(n+1)}{x}\frac{dZ_n}{dx} - \frac{(n+1)n}{x^2}Z_n = -Z_n$$

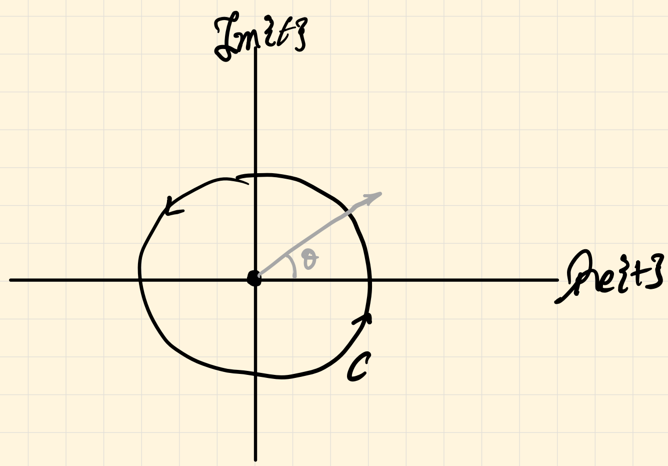
$$\frac{d^2Z_n}{dx^2} + \frac{1}{x}\frac{dZ_n}{dx}[\cancel{-n}\cancel{+n}+1] + \frac{Z_n}{x^2}[\cancel{1}-n\cancel{-1}]n = -Z_n$$

$$\left[Z_n'' + \frac{1}{x}Z_n' - \frac{Z_n}{x^2}n^2 = -Z_n\right]x^2$$

$$x^2 Z_n'' + x Z_n' - Z_n n^2 + Z_n x^2 = 0 \quad \checkmark$$

Copy of the example in the lecture notes

> **Example: Schläfli integral for non-polynomials and its relation to generating functions**. The Bessel functions of the first kind are not polynomials, but they have a generating function $g(x,t) = e^{(x/2)(t-1/t)} = \sum_n J_n(x)t^n$ (see table). If we apply the residue theorem to an integral containing the generating function, we get $\oint_C \frac{e^{(x/2)(t-1/t)}}{t^{n+1}}dt = \oint_C \sum_m J_m(x)t^{m-n-1} = 2\pi i J_n(x)$ with C encircling $t=0$. With the change of variable $t = e^{i\theta} \implies e^{(x/2)(t-1/t)} = e^{ix\sin\theta}$, and C being the unit circle, we find $2\pi i J_n(x) = \int_0^{2\pi} e^{i(x\sin\theta - n\theta)} i d\theta$ in general for $x \in \mathbb{C}$ or $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)d\theta$ for $x \in \mathbb{R}$. In this case, the generating function was used to get (a) Schläfli representation, but we can also derive generating functions (for example, for polynomials) using the/a Schläfli representation.



Problem. In a diffraction problem for a circular apperture, the integral

$$\Phi = \int_0^a \int_0^{2\pi} e^{ibr\cos\theta}\, d\theta \;\, r\,dr$$

appears, where a, b are constants. Evaluate the integral using Bessel's functions.

The integral representation of the Bessel functions is

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta \quad \text{or} \quad J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)} d\theta$$

for $n=0$,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta)\, d\theta =$$

$\cos(x\sin\theta)$ repeats itself in all four quadrants

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\, d\theta$$

because of Euler's formula and $\int_0^{2\pi} \sin(x\sin\theta)\, d\theta = 0$

then

$$\Phi = 2\pi \int_0^a J_0(br)\, r\,dr$$

using the recurrence formula $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ with $n=1$,

$$\Phi = \frac{2\pi a b}{b^2} J_1(ab).$$

Problem. Evaluate the integral

$$\int_0^{\infty} e^{-ax} J_0(bx)\, dx, \qquad a>0,\; b>0.$$

Solution. Introduce the integral representation of the Bessel function

$$= \int_0^{\infty} e^{-ax}\, dx \;\; \frac{2}{\pi}\int_0^{\pi/2} \cos(bx\sin\varphi)\, d\varphi$$

exchanging the integration order

$$= \frac{2}{\pi}\int_0^{\pi/2} d\varphi \int_0^{\infty} e^{-ax}\cos(bx\sin\varphi)\, dx$$

integrating,

$$= \frac{2}{\pi}\int_0^{\pi/2} d\varphi \;\frac{a}{a^2 + b^2\sin^2\varphi} = \frac{1}{\sqrt{a^2+b^2}} \quad \blacksquare$$

Problem. John Bernoulli needed to solve the following non-linear differential equation:

$$y' = x^2 + y^2$$

but he couldn't. His brother and rival James Bernoulli could solve it by transforming it into a linear second order ODE with the change of variable

$$y = -\frac{1}{u}\frac{du}{dx}$$

into

$$\Rightarrow \quad u'' + x^2 u = 0$$

Solve the equation for u(x) by further transforming it into a Bessel equation.
Solution:

Generalized Bessel ODE

$$u'' + \underbrace{\frac{1-2a}{x}}_{=0} u' + \left[(bcx^{c-1})^2 + \underbrace{\frac{a^2 - p^2c^2}{x^2}}_{=0}\right] u = 0$$

we want:

$$\Rightarrow 1 - 2a = 0$$
$$\Rightarrow a = 1/2$$

$$\Rightarrow \left(\tfrac{1}{2}\right)^2 - p^2c^2 = 0$$
$$\Rightarrow p^2c^2 = \frac{1}{4}$$

$= x^2$
we need $c = 2$
$\rightarrow (2b\,x)^2$
we need $b = 1/2$
$\rightarrow x^2$ ✓

$$p^2 \cdot 4 = \frac{1}{4}$$
$$\Rightarrow p^2 = \frac{1}{16} \Rightarrow p = \pm 4$$

$$\Rightarrow u(x) = x^a Z_p(bx^c) = \sqrt{x}\, Z_4\left(\frac{x^2}{2}\right)$$
$$u(x) = c_1\sqrt{x}\, J_4(x^2/2) + c_2\sqrt{x}\, Y_4(x^2/2) \quad \blacksquare$$

Problem. The solutions of the second order differential equation

$$u'' - xu = 0$$

are called the Airy functions. Solve the differential equation by first transforming it into a generalized Bessel ODE.
Solution:

Generalized Bessel ODE

$$u'' + \underbrace{\frac{1-2a}{x}}_{\text{we want }=0} u' + \left[(bcx^{c-1})^2 + \underbrace{\frac{a^2 - p^2c^2}{x^2}}_{\text{we want }=0} \right] u = 0$$

$\Rightarrow a = 1/2$

$\Rightarrow p^2c^2 = 1/4$

$p^2 \cdot \frac{9}{4} = \frac{1}{4}$

$\Rightarrow p = \pm 1/3$

we want $= -x$

$\Rightarrow c = 3/2$

$\Rightarrow \left(b \frac{3}{2} x^{1/2}\right)^2 \overset{!}{=} -x$

$\Rightarrow b^2 \cdot \left(\frac{3}{2}\right)^2 x = -x$

$\Rightarrow b^2 \cdot \left(\frac{3}{2}\right)^2 = -1 \Rightarrow b = \frac{2}{3} i$

$$\Rightarrow u(x) = x^a Z_p(bx^c) = x^{1/2} Z_{1/3}\left(\frac{2}{3} i\, x^{3/2}\right)$$

in terms of the Modified Bessel functions (Bessel of imaginary argument)

$$u(x) := c_1 Ai(x) + c_2 Bi(x)$$

$$Ai(x) = \frac{x^{1/2}}{3}\left[I_{-1/3}\left(\frac{2x^{3/2}}{3}\right) - I_{1/3}\left(\frac{2x^{3/2}}{3}\right) \right] = \frac{1}{\pi}\left(\frac{x}{3}\right)^{1/2} K_{1/3}\left(\frac{2x^{3/2}}{3}\right)$$

$$Bi(x) = \left(\frac{x}{3}\right)^{1/2} \left[I_{-1/3}\left(\frac{2x^{3/2}}{3}\right) + I_{1/3}\left(\frac{2x^{3/2}}{3}\right) \right]$$

Problem: show that $j_0(x) = \frac{\sin x}{x}$

starting from the series expansion of $J_{1/2}(x)$.

$$\boxed{\begin{array}{l}\Gamma(1/2) = \sqrt{\pi} \\ \Gamma(n+1) = n! \\ \Gamma(n+1)\ n\Gamma(n)\end{array}}$$

Solution:

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3/2)} \left(\frac{x}{2}\right)^{2n+1/2}$$

we know that

$$\sin x = \underbrace{x}_{n=0} - \underbrace{\frac{x^3}{3!}}_{n=1} + \underbrace{\frac{x^5}{5!}}_{n=2} - \underbrace{\frac{x^7}{7!}}_{n=3} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{\underbrace{(2n+1)}_{m}!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{\Gamma(\underbrace{2n+2}_{m+1})}$$

$$2^{2n+1/2} = 2^{\underbrace{(2n+2)}_{2(n+1)}-1-1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n+1+1/2)} (-1)^n \frac{x^{2n+1-1/2}}{2^{2(n+1)-1} \cdot 2^{-1/2}}$$

$$\boxed{2^{2x-1}\Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\ \Gamma(2x)}$$ Tables
Gamma function duplication formula

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}\,\Gamma(2n+2)} (-1)^n x^{2n+1} \cdot \frac{x^{-1/2}}{\underbrace{2^{-1/2}}_{\text{doesn't depend on } n}}$$

$\sin(x)$

$$= \sqrt{\frac{2}{\pi}}\, x^{-1/2} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

finally,

$$\Rightarrow j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x)$$
$$= \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \sin x$$
$$= \frac{\sin x}{x}$$

Problem. Show that the Ansatz $\psi(z) = e^{\frac{-z^2}{2}} v(z)$ on the differential equation

$$\frac{d^2}{dz^2}\psi(z) + \left(\frac{2E}{\hbar\omega} - z^2\right)\psi(z) = 0 \qquad (1)$$

with $E, \hbar, \omega$ being constants, transforms it into the Hermite ODE.
Solution:

$$\psi(z) = e^{-z^2/2} v(z) \quad (2)$$

$$\frac{d\psi}{dz} = -z e^{-z^2/2} v(z) + e^{-\frac{z^2}{2}}\frac{dv(z)}{dz} = e^{-z^2/2}\left(-zv + \frac{dv}{dz}\right)$$

$$\frac{d^2\psi}{dz^2} = -ze^{-z^2/2}\left(-zv + \frac{dv}{dz}\right) + e^{-z^2/2}\left(-v - z\frac{dv}{dz} + \frac{d^2v}{dz^2}\right)$$

$$= e^{-z^2/2}\left(z^2v - 2z\frac{dv}{dz} - v + \frac{d^2v}{dz^2}\right) \qquad (3)$$

Substituting (3), (2) in (1), we see that the factor $e^{-z^2/2}$ cancels out:

$$\cancel{z^2v} - 2z\frac{dv}{dz} - v + \frac{d^2v}{dz^2} + \frac{2E}{\hbar\omega}v - \cancel{z^2v} = 0$$

$$\Rightarrow v'' - 2zv' + \left(\frac{2E}{\hbar\omega} - 1\right)v = 0 \quad ✓$$

Problem: Show the recurrence formula

$$H_n'(x) = 2n H_{n-1}(x)$$

Solution:

We start from the generating function

$$\Phi(x,h) = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!} = e^{-h^2+2hx}$$

And try

$$\frac{\partial\Phi}{\partial x} = 2he^{-h^2+2hx} = 2h\Phi = 2h$$

we substitute Φ and $\frac{\partial\Phi}{\partial x}$ by the series.

$$\Phi(x,h) = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!}; \quad \frac{\partial\Phi}{\partial x} = \sum_{n=0}^{\infty} H_n'(x)\frac{h^n}{n!}$$

$$\sum_{n=0}^{\infty} H_n'(x)\frac{h^n}{n!} = \sum_{n=0}^{\infty} 2h\, H_n(x)\frac{h^n}{n!}$$

$$2\sum_{n=0}^{\infty} H_n(x)\frac{h^{n+1}}{n!}$$

let $m = n+1 \Rightarrow$ when $n=0$, $m=1$

$$\sum_{n=0}^{\infty} H_n'(x)\frac{h^n}{n!} = 2\sum_{m=1}^{\infty} H_{m-1}(x)\frac{h^m}{(m-1)!}$$

$m=n$, recursivity of factorial

$$\sum_{m=0}^{\infty} H_m'(x)\frac{h^m}{m(m-1)!} = 2\sum_{m=1}^{\infty} H_{m-1}(x)\frac{h^m}{(m-1)!}$$

Finally, for $m \geq 1$, $\frac{H_m'(x)}{m} = 2H_{m-1}(x) \Rightarrow H_n'(x) = 2n H_{n-1}(x)$ ✓

Problem: Write an integral representation of the Hermite polynomials.

Solution:

Start from the generating function

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

and multiply by t^{-m-1}:

$$t^{-m-1}e^{-t^2+2tx} = \sum_{n=0}^{\infty} t^{-m-1+n}\frac{H_n(x)}{n!}$$

integrate in the complex plane around the origin:

$$\oint t^{-m-1}e^{-t^2+2tx}\,dt = \oint \sum_{n=0}^{\infty} t^{-m-1+n}\frac{H_n(x)}{n!}\,dt$$

With poles at $t=0$, we find with the residue theorem that only the term $n=m$ survives:

$$\oint t^{-m-1}e^{-t^2+2tx}\,dt = 2\pi i\,\frac{H_m(x)}{m!}$$

$$\Rightarrow H_m(x) = \frac{m!}{2\pi i}\oint t^{-m-1}e^{-t^2+2tx}\,dt$$

Problem: find the normalization of the Hermite polynomials.

Solution:

We start this time from the generating function (we multiply two of them, one with the variable t and one with the variable s) and build the following expression

$$e^{-x^2}e^{-s^2+2sx}e^{-t^2+2tx} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-x^2}H_m(x)H_n(x)\frac{s^m t^n}{m!\,n!}$$

now we integrate over the interval $]-\infty,\infty[$ and from the ortogonality, we know the terms other than $m=n$ will be zero.

$$\sum_{n=0}^{\infty}\frac{(st)^n}{n!\,n!}\int_{-\infty}^{\infty} e^{-x^2}[H_n(x)]^2\,dx = \int_{-\infty}^{\infty} e^{-x^2-s^2+2sx-t^2+2tx-2st+2st}\,dx$$

$$= \int_{-\infty}^{\infty} e^{\underbrace{-(x-s-t)^2}_{\text{Known integral}}}e^{2st}\,dx = \sqrt{\pi}\,e^{2st} = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^n(st)^n}{n!}$$

Examing coefficients term by term of the power st we obtain

$$\int_{-\infty}^{\infty} e^{-x^2}[H_n(x)]^2\,dx = 2^n\sqrt{\pi}\,n!$$