

Problem: Find all the spherical harmonics for $\ell=1$

Solution:

From the definition, we know that $-\ell \leq m \leq \ell$.

This means we need

$$\ell=1$$

$$m=-1, 0, 1$$

Then,

$$Y_{1,1}(\theta, \varphi) \propto P_1^1(\cos\theta) e^{i\varphi}$$

(normalization:

$$\sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}} = \sqrt{\frac{3}{4\pi} \cdot \frac{0!}{2!}} = \sqrt{\frac{3}{8\pi}}$$

$$= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}$$

$$Y_{1,0}(\theta, \varphi) \propto P_1^0(\cos\theta) e^0$$

(normalization = $\sqrt{\frac{3}{4\pi} \cdot \frac{1!}{1!}} = \sqrt{\frac{3}{4\pi}}$

$$= \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,-1}(\theta, \varphi) \propto P_1^{-1}(\cos\theta) e^{-i\varphi}$$

We know that

$$Y_{\ell-m} = (-1)^m \overline{Y_{\ell m}}$$

$$= -\overline{Y_{1,1}} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\varphi}$$

comes from properties of assoc. Legendre and can be shown by Rodrigues's formula

Problem: Write the functions $f(\theta) = \cos\theta$, $g(\theta) = \sin\theta$

in terms of the spherical harmonics.

► $f(\theta)$ is the Legendre polynomial $P_1(\cos\theta) = \cos\theta$.

We try by inspection:

- with $\ell=0$, $m=0$

$$Y_{0,0}(\theta, \varphi) \propto P_0(\cos\theta) e^0 = 1 \quad \times \quad \text{for } f(\theta)$$

• with $l=1$, $m=-1, 0, 1$

$$Y_{1,0}(\theta, \varphi) \propto \cos \theta \quad \checkmark$$

for $f(\theta)$

$$Y_{1,1}(\theta, \varphi) \propto \sin \theta \quad \times$$

$$\text{So, } f(\theta) = \underbrace{\left(\sqrt{\frac{4\pi}{3}}\right)}_{\text{coeff}} \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{4\pi}{3}} Y_{1,0}(\theta, \varphi)$$

- For higher order l sph. harmonics, and $m=0$, we get a higher order Legendre polynomial $\Rightarrow$ by orthogonality, $l > 1$ won't work.
- For higher order l and $m \neq 0$, none will work because of the φ dependency ($e^{im\varphi}$)

✦ for $g(\theta)$, we see that

$$Y_{1,0}(\theta, \varphi) \propto \cos \theta \quad \times$$

$$Y_{1,1}(\theta, \varphi) \propto \sin \theta e^{i\varphi} \quad \times$$

$$Y_{1,-1}(\theta, \varphi) \propto \sin \theta e^{-i\varphi} \quad \times$$

Then this means we must expand in this case as

$$g(\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^l g_{lm} Y_{lm}(\theta, \varphi)$$

but we know that none of the $m \neq 0$ will work because there is no φ dependency. This means

$$g(\theta) = \sum_{l=0}^{\infty} g_{l0} Y_{l0}(\theta, \varphi)$$

which is equivalent to the problem of expanding $\sin \theta$ as a series of Leg. polyn. eval. on $\cos \theta$, that is

$$\sin \theta = \sum_{l=0}^{\infty} \tilde{g}_l P_l(\cos \theta)$$

Note how different the solution becomes by just rotating the coordinate θ . For solving problems, one should exploit the symmetries so that the solution is as simple as possible.